Warm Up: Evaluate each limit. Note the strategy you are using to evaluate the limit.

1. $\lim_{x\to 2}\frac{x^2+x-6}{x-2} = \lim_{x\to 2}\frac{(x+3)\cancel{(x-2)}}{\cancel{x-2}}$ $= 2+3$ $= 5$

$\downarrow$

$\frac{2^2+2-6}{2-2}=\frac{0}{0}$

$\downarrow$

Indeterminate
factor + reduce

2. $\lim_{x\to\infty}\frac{3x}{x^2-3} = 0$

Indeterminate

$\downarrow$

End Behavior Model : $\frac{3x}{x^2}=\frac{3}{x}$

As $x\to\infty$, EBM $\to 0$

3. $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$\downarrow$

$\frac{\sin 0}{0}=\frac{0}{0}$

Indeterminate... use common limit

4. $\lim_{x\to 0}\frac{e^{2x}-1}{x} = 2$

$\downarrow$

$\frac{e^{2(0)}-1}{0}=\frac{e^0-1}{0}=\frac{0}{0}$

Indeterminate... no alg. strategies; graph

In each example above, the limit takes on an indeterminate form (either $\frac{0}{0}$ or $\frac{\infty}{\infty}$). Notice that only the first example was evaluated using algebraic techniques. Now that we know how to find a derivative, we have an alternate way to evaluate limits such as these using the following rule:

L'Hôpital's (aka L'Hospital's) Rule * Notecard *

Suppose that f and g are differentiable functions on the open interval I containing a, but not necessarily at $x = a$. Also suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are both 0 or are both $\pm\infty$ and $g'(x)\neq 0$ on I if $x\neq a$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ when the latter exists.}$$

In other words, if a limit yields the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then we can find the limit of the quotient of the derivatives of the functions to evaluate that limit.

Remember to check all three parts of the hypothesis BEFORE applying this rule!

① f & g differentiable near x=a
② limits of f & g exist and both 0 or both inf.
③ limit of f' & g' must exist

Example 1: Identify the indeterminate form for each limit. Then, use L'Hôpital's Rule to evaluate.

a) $\lim_{x\to 2}\frac{x^2+x-6}{x-2} \overset{\text{l'hop}}{=} \lim_{x\to 2}\frac{2x+1}{1}$

Indet: $\frac{0}{0}$

$= 2(2)+1$
$= 5$

b) $\lim_{x\to\infty}\frac{3x}{x^2-3} \overset{\text{l'hop}}{=} \lim_{x\to\infty}\frac{3}{2x} = 0$

Indet: $\frac{\infty}{\infty}$

As $x\to\infty$, $\frac{3}{2x}\to 0$

c) $\lim_{x\to 0}\frac{\sin x}{x} \overset{l'hop}{=} \lim_{x\to 0}\frac{\cos x}{1}$

Indet: $\frac{0}{0}$

$= \cos 0$

$= 1$

d) $\lim_{x\to 0}\frac{e^{2x}-1}{x} \overset{l'hop}{=} \lim_{x\to 0}\frac{2e^{2x}}{1}$

Indet: $\frac{0}{0}$

$= 2e^{2(0)}$

$= 2e^0$

$= 2$

Example 2: Determine whether or not L'Hopsital's Rule applies. If so, use it to evaluate the limit. If not, explain why not.

a) $\lim_{x\to\infty}\frac{e^{-x}}{x}$

As $x\to\infty$, $e^{-x}\to 0$ and as $x\to\infty$, denom $\to\infty$. $\therefore$ L'Hôpital's does not apply.

b) $\lim_{x\to\infty}\frac{x-\sin x}{x+\sin x} \overset{l'hop.}{=} \lim_{x\to\infty}\frac{1-\cos x}{1+\cos x}$

Indet: $\frac{\infty}{\infty}$ ... try l'hop.

BUT... $\left.\begin{array}{l}\lim_{x\to\infty} 1-\cos x \text{ DNE} \\ \text{and} \\ \lim_{x\to\infty} 1+\cos x \text{ DNE}\end{array}\right\}$ $\therefore$ we cannot use l'hopital's

Just For Fun ... the above limits CAN be found...

a) $\lim_{x\to\infty}\frac{e^{-x}}{x} = 0$

As $x\to\infty$, $e^{-x}\to$ very small #

As $x\to\infty$, denom. grows faster than that very small #.

$\therefore$ limit is zero

b) $\lim_{x\to\infty}\frac{x-\sin x}{x+\sin x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim_{x\to\infty}\frac{1-\frac{\sin x}{x}}{1+\frac{\sin x}{x}}$

As $x\to\infty$, $\frac{\sin x}{x}\to 0$.

and $\frac{1-0}{1+0} = 1$ $\therefore$ limit $= 1$