AP Calculus

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Block: ______

Chapter 7 Test Review Solutions

For questions 1 – 21, integrate each of the following indefinite integrals.

1. $\int \frac{3x}{\sqrt[3]{x^2+3}}\,dx = 3\int u^{-\frac{1}{3}} \cdot \frac{1}{2}du$

Let $u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \frac{3}{2}\cdot\frac{3}{2}u^{2/3} + C$

$= \boxed{\frac{9}{4}(x^2+3)^{2/3} + C}$

3. $\int \frac{x-1}{x+1}\,dx = \int\left(1 - \frac{2}{x+1}\right)dx$

$$\begin{array}{r} 1 \\ x+1\overline{)\,x-1} \\ -(x+1) \\ \hline -2 \end{array}$$

$= \int 1\,dx - 2\int\frac{1}{x+1}dx$

$= \boxed{x - 2\ln|x+1| + C}$

4. $\int 9\sin x\,dx = 9\int \sin x\,dx$

$= \boxed{-9\cos x + C}$

5. $\int 7^{3x}\,dx = \int 7^u \cdot \frac{1}{3}du$

Let $u = 3x$

$du = 3dx$

$\frac{1}{3}du = dx$

$= \frac{1}{3}\int 7^u\,du$

$= \frac{1}{3}\cdot\frac{7^u}{\ln 7} + C$

$= \boxed{\frac{7^{3x}}{3\ln 7} + C}$

6. $\int 2\tan(x)\,dx = 2\int \tan(x)\,dx$

$= \boxed{-2\ln|\cos x| + C}$

8. $\int \frac{(x+1)^2}{x^{1/3}}\,dx$

$= \int \frac{x^2+2x+1}{x^{1/3}}\,dx$

$= \int\left(x^{5/3} + 2x^{2/3} + x^{-1/3}\right)dx$

$\boxed{\frac{3}{8}x^{8/3} + \frac{6}{5}x^{5/3} + \frac{3}{2}x^{2/3} + C}$

9. $\int \cos^2(8x)\,dx$

Let $u = 8x$

$du = 8dx$

$\frac{1}{8}du = dx$

$= \int \cos^2 u \cdot \frac{1}{8}du$

$= \frac{1}{8}\left[\frac{1}{2}u + \frac{1}{4}\sin(2u) + C\right]$

$= \frac{1}{8}\cdot\frac{1}{2}\cdot 8x + \frac{1}{8}\cdot\frac{1}{4}\sin(2\cdot 8x) + \frac{1}{8}C$

$= \boxed{\frac{1}{2}x + \frac{1}{32}\sin(16x) + C}$

10. $\int \frac{dx}{4+9x^2}$

$= \int \frac{dx}{4\left(1+\frac{9x^2}{4}\right)}$

$= \frac{1}{4}\int\frac{dx}{1+\left(\frac{3}{2}x\right)^2}$

Let $u = \frac{3x}{2}$; $du = \frac{3}{2}dx$

$\frac{1}{4}\int\frac{1}{1+u^2}\cdot\frac{2}{3}du$

$= \frac{1}{6}\tan^{-1}u + C$

$= \boxed{\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C}$

11. $\int\left(10\cos t + \sin^2(10t)\right)dt$

$= \int 10\cos t\,dt + \int \sin^2(10t)\,dt$

Let $u = 10t$

$du = 10dt$; $\frac{1}{10}du = dt$

$= 10\sin t + \frac{1}{10}\int \sin^2 u\,du$

$= 10\sin t + \frac{1}{10}\left(\frac{1}{2}u - \frac{1}{4}\sin(2u) + C\right)$

$= \boxed{10\sin t + \frac{1}{2}t - \frac{1}{40}\sin(20t) + C}$

12. $\int \sec^2 x\,dx$

$= \boxed{\tan x + C}$

14. $\int 5e^{3x}\cot\left(e^{3x}\right)dx$

$5\int\frac{e^{3x}\cos(e^{3x})}{\sin(e^{3x})}\,dx$

Let $u = \sin(e^{3x})$

$du = \cos(e^{3x})\cdot e^{3x}\cdot 3dx$

$\frac{1}{3}du = e^{3x}\cos(e^{3x})\,dx$

$= \frac{5}{3}\int\frac{du}{u} = \frac{5}{3}\ln|u| + C$

$= \boxed{\frac{5}{3}\ln\left|\sin(e^{3x})\right| + C}$

15. $\int \frac{1}{\sec(12x)}\,dx = \int \cos(12x)\,dx$

Let $u = 12x$

$du = 12dx$

$\frac{1}{12}du = dx$

$= \int \cos u \cdot \frac{1}{12}du$

$= \frac{1}{12}\sin u + C$

$= \boxed{\frac{1}{12}\sin(12x) + C}$

17. $\int \tan^2\left(\frac{x}{5}\right) dx$

Let $u = \frac{x}{5}$, $du = \frac{1}{5}dx$, $5du = dx$

$= \int \tan^2 u \cdot 5du$

$= 5[\tan u - u + C]$

$= 5[\tan(\frac{x}{5}) - (\frac{x}{5}) + C]$

or $5\tan(\frac{x}{5}) - x + C$

18. $\int \frac{x}{x^2-4} dx$

Let $u = x^2-4$, $du = 2x\,dx$, $\frac{1}{2}du = 2dx$

$= \int \frac{\frac{1}{2}du}{u}$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2-4| + C$

19. $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}} dx$

$\left.\begin{array}{l}\text{Let } u = e^x \\ du = e^x dx\end{array}\right\} = \int \frac{1}{\sqrt{1-u^2}} du$

$= \sin^{-1} u + C$

$= \sin^{-1}(e^x) + C$

20. $\int \csc^2 x\, dx$

$= -\cot x + C$

21. $\int xe^{-x^2+3} dx$

Let $u = -x^2+3$, $du = -2x\,dx$, $-\frac{1}{2}du = x\,dx$

$= \int e^u \cdot -\frac{1}{2}du$

$= -\frac{1}{2}e^u + C$

$= -\frac{1}{2}e^{(-x^2+3)} + C$

For questions 22 – 26, evaluate each definite integral without a calculator. Check your answer with your calculator.

22. $\int_{-1}^{1}(x^2-5)^2 dx = \int_{-1}^{1}(x^4 - 10x^2 + 25)\, dx$

$= \left.\frac{x^5}{5} - \frac{10x^3}{3} + 25x\right|_{-1}^{1}$

$= \left[\frac{1^5}{5} - \frac{10(1)^3}{3} + 25(1)\right] - \left[\frac{(-1)^5}{5} - \frac{10(-1)^3}{3} + 25(-1)\right]$

$= \left[\frac{1}{5} - \frac{10}{3} + 25\right] - \left[-\frac{1}{5} + \frac{10}{3} - 25\right] = \frac{1}{5} - \frac{10}{3} + 25 + \frac{1}{5} - \frac{10}{3} + 25$

$= \frac{2}{5} - \frac{20}{3} + 50 = \frac{6 - 100 + 750}{15} = \boxed{\frac{656}{15}}$

23. $\int_{-4}^{-2} \frac{dx}{x^2+6x+10} = \int_{-4}^{-2} \frac{dx}{x^2+6x+9+10-9}$

$= \int_{-4}^{-2} \frac{dx}{(x+3)^2+1}$

Let $u = x+3$, $du = dx$

When $x=-4, u=-1$; $x=-2, u=1$

$\int_{-1}^{1} \frac{du}{u^2+1}$

$= \left.\tan^{-1} u\right|_{-1}^{1}$

$= \tan^{-1}(1) - \tan^{-1}(-1)$

$= \frac{\pi}{4} - (-\frac{\pi}{4}) = \boxed{\frac{\pi}{2}}$

24. $\int_0^1 \frac{3+x}{x^2+1} dx \to \int_0^1 \frac{3}{x^2+1} dx + \int_0^1 \frac{x}{x^2+1} dx$

$= \left.3\tan^{-1}x\right|_0^1 + \frac{1}{2}\int_1^2 \frac{du}{u}$

Let $u = x^2+1$, $du = 2x\,dx$, $\frac{1}{2}du = x\,dx$

When $x=0, u=1$; $x=1, u=2$

$= \left.3\tan^{-1}x\right|_0^1 + \left.\frac{1}{2}\ln|u|\right|_1^2$

$= [3\tan^{-1}(1) - 3\tan^{-1}(0)] + [\frac{1}{2}\ln|2| - \frac{1}{2}\ln|1|]$

$= \boxed{3(\frac{\pi}{4}) + \frac{1}{2}\ln 2}$

25. $\int_0^6 \frac{dx}{7-x}$

Let $u = 7-x$, $du = -dx$, $-du = dx$

When $x=0, u=7$; $x=6, u=1$

$= \int_7^1 \frac{-du}{u}$

$= \left.-\ln|u|\right|_7^1$

$= -\ln(1) - (-\ln(7))$

$= \boxed{\ln 7}$

26. $\int_4^9 \sqrt{x}(3-4x)\, dx = \int_4^9 x^{1/2}(3-4x)\, dx$

$= \int_4^9 (3x^{1/2} - 4x^{3/2})\, dx$

$= \left.\left[3 \cdot \frac{2}{3}x^{3/2} - 4 \cdot \frac{2}{5}x^{5/2}\right]\right|_4^9$

$= \left.\left(2x^{3/2} - \frac{8}{5}x^{5/2}\right)\right|_4^9$

$\left[2(9)^{3/2} - \frac{8}{5}(9)^{5/2}\right] - \left[2(4)^{3/2} - \frac{8}{5}(4)^{5/2}\right]$

$\left[2(27) - \frac{8}{5}(243)\right] - \left[2(8) - \frac{8}{5}(32)\right]$

$\left(54 - \frac{1944}{5}\right) - \left(16 - \frac{256}{5}\right)$

$38 - \frac{1688}{5}$

$\frac{190 - 1688}{5} = \boxed{\frac{-1498}{5}}$

27. Derive (SHOW EVERY STEP) $y = y_0 e^{kt}$ from $\frac{dy}{dt} = ky$ and $y(0) = y_0$.

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int k \cdot dt$$

$$\ln|y| = kt + C$$

$$|y| = e^{kt+C}$$

$$|y| = e^{kt} \cdot \underbrace{e^C}_{\text{another const.}}$$

$$|y| = Ce^{kt}$$

We don't know the sign of initial condition, so...

$$y = \pm Ce^{kt}$$

If $y(0) = y_0$, then $y_0 = \pm Ce^{k(0)}$

$y_0 = \pm C$

$\therefore y = y_0 e^{kt}$

28. Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\frac{3x^2+1}{2y} = \frac{dy}{dx}$

a) Find the slope of the graph of f at the point where $x = 1$.

When $x=1$, $y=4$.

$$\left.\frac{dy}{dx}\right|_{(1,4)} = \frac{3(1)^2+1}{2(4)} = \frac{4}{8} = \frac{1}{2}$$

Slope only defined when y is on $(-\infty, 0) \cup (0, \infty)$.

b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$

pt: $(1,4)$

slope: $\frac{1}{2}$

$\boxed{y - 4 = \frac{1}{2}(x-1)}$

or $f(x) = \frac{1}{2}(x-1) + 4$

$f(1.2) = \frac{1}{2}(1.2-1) + 4$

$= \frac{1}{2}(.2) + 4$

$\boxed{f(1.2) \approx 4.1}$

c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2+1}{2y}$ with the initial condition $f(1) = 4$.

$$\int 2y\, dy = \int (3x^2+1)\, dx$$

$$2 \cdot \frac{y^2}{2} = 3 \cdot \frac{x^3}{3} + x + C$$

$$y^2 = x^3 + x + C$$

When $x=1$, $y=4$

$4^2 = 1^3 + 1 + C$

$16 = 2 + C$

$14 = C$

$\therefore y^2 = x^3 + x + 14$

and $\sqrt{y^2} = \sqrt{x^3+x+14}$

$|y| = \sqrt{x^3+x+14}$

$\boxed{f(x) = \sqrt{x^3+x+14}}$

Initial cond. tells us y is on $(0,\infty)$, So $|y| = y$

d) Use your solution from part c to find $f(1.2)$

$$f(1.2) = \sqrt{(1.2)^3 + 1.2 + 14}$$

$$= \sqrt{16.928}$$

$$\approx 4.114$$

Close to approx. from part b :)

29. If $\frac{dy}{dx} = y\sec^2 x$ and $y = 5$ when $x = 0$, then $y =$

A $e^{\tan x} + 4$

B $e^{\tan x} + 5$

C $\boxed{5e^{\tan x}}$

D $\tan x + 5$

E $\tan x + 5e^x$

$$\frac{dy}{dx} = y\sec^2 x$$

$$\int \frac{dy}{y} = \int \sec^2 x\, dx$$

$$\ln|y| = \tan x + C$$

y is on $(-\infty,0) \cup (0,\infty)$ but initial cond $y=5$... use $(0,\infty)$. $|y| = y$

$$y = e^{\tan x + C}$$

$$y = e^{\tan x} \cdot e^C$$

$$y = Ce^{\tan x}$$

$$5 = Ce^{\tan(0)}$$

$$5 = C$$

$\therefore y = 5e^{\tan x}$

30. [No Calculator] If $\frac{dy}{dt} = -2y$ and if $y = 1$ when $t = 0$, what is the value of t for which $y = \frac{1}{2}$?

A) $-\frac{1}{2}\ln 2$

B) $-\frac{1}{4}$

C) $\boxed{\frac{1}{2}\ln 2}$

D) $\frac{\sqrt{2}}{2}$

E) $\ln 2$

$\int \frac{dy}{y} = \int -2dt$

y is on $(-\infty, 0) \cup (0, \infty)$ but $y = 1$ ... use $(0, \infty)$ So $|y| = y$.

$\ln|y| = -2t + C$

$|y| = e^{-2t+C}$

$y = e^{-2t} \cdot e^{C}$

$y = Ce^{-2t}$

If $t = 0$ & $y = 1$, then $1 = Ce^{-2(0)}$

$1 = C$

$\therefore y = 1e^{-2t}$

when $y = \frac{1}{2}$...

$\frac{1}{2} = e^{-2t}$

$\ln(\frac{1}{2}) = \ln e^{-2t}$

$\ln(\frac{1}{2}) = -2t$

$-\frac{1}{2}\ln(\frac{1}{2}) = t$

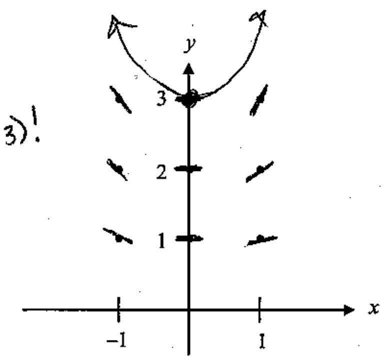
$-\frac{1}{2}\ln(2^{-1}) = \frac{1}{2}\ln 2 = t$

31. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.

b) Draw a particular solution if $\boxed{f(0) = 3}$

Be sure to include $(0, 3)$!



pt	slope: $\frac{xy}{2}$
$(-1,1)$	$-\frac{1}{2}$
$(-1,2)$	-1
$(-1,3)$	$-\frac{3}{2}$
$(0,1)$	0
$(0,2)$	0
$(0,3)$	0
$(1,1)$	$\frac{1}{2}$
$(1,2)$	1
$(1,3)$	$\frac{3}{2}$

c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

$\frac{dy}{dx} = \frac{xy}{2}$

$\int \frac{dy}{y} = \int \frac{x}{2} dx$

y is on $\cancel{(-\infty, 0)}$ or $(0, \infty)$. initial cond: $f(0) = 3$ $\therefore |y| = y$

$\ln|y| = \frac{1}{2} \cdot \frac{x^2}{2} + C$

$\ln|y| = \frac{x^2}{4} + C$

$y = e^{x^2/4 + C}$

$y = e^{\frac{x^2}{4}} \cdot e^{C}$

$y = Ce^{\frac{x^2}{4}}$

When $x = 0$, $y = 3$.

$3 = Ce^{0}$

$3 = C$

$\boxed{\therefore y = 3e^{\frac{x^2}{4}}}$

$y = f(x)$

$f(0.2) = 3e^{\frac{(0.2)^2}{4}}$

$f(0.2) \approx 3.030$

32. [Calculator] During a certain epidemic, the number of people that are infected at any time increases at rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?

When $t = 0$, $N = 1000$

Let N = # people infected

$\frac{dN}{dt} = k \cdot N$

So $N = N_0 e^{kt}$, $N_0 = 1000$

$N = 1000e^{kt}$

When $t = 7$, $N = 1200$

$1200 = 1000e^{7k}$

$1.2 = e^{7k}$

$\ln(1.2) = 7k$

$\frac{\ln(1.2)}{7} = k$

$\therefore N = 1000e^{\frac{1}{7}\ln(1.2) \cdot t}$

When $t = 12$,

$N = 1000e^{\frac{1}{7}\ln(1.2) \cdot 12}$

$N \approx 1366.90798$

≈ 1367 people were infected when epidemic was discovered

33. [Calculator] Population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 10 years, then the value of k is

$y = y_0 e^{kt}$

initial pop $= y_0$
So when $t = 10$, $y = 2y_0$

$$2y_0 = y_0 e^{k(10)}$$
$$2 = e^{10k}$$
$$\ln(2) = 10k$$

$\therefore k = \frac{1}{10}\ln(2)$

34. [No Calculator] Bacteria in a certain culture increase at rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

Let B = # bacteria present

$$\frac{dB}{dt} = k \cdot B$$

$$B = B_0 e^{kt}$$

When $t = 3$, $B = 2B_0$

$$2B_0 = B_0 e^{k(3)}$$
$$2 = e^{3k}$$
$$\ln(2) = 3k$$
$$\tfrac{1}{3}\ln(2) = k$$

$\therefore B = B_0 e^{\frac{1}{3}\ln(2) \cdot t}$

Find t when $B = 3B_0$

$$3B_0 = B_0 e^{\frac{1}{3}\ln(2)\cdot t}$$
$$3 = e^{\frac{1}{3}\ln(2)\cdot t}$$
$$\ln(3) = \tfrac{1}{3}\ln(2) \cdot t$$
$$\boxed{\frac{3\ln(3)}{\ln(2)} = t}$$

← Non-Calc answer

If curious, $t \approx 4.755$ hr (calc ans)

35. Solutions to the differential equation $\frac{dy}{dx} = xy^3$ will also satisfy $\frac{d^2y}{dx^2} = y^3(1+3x^2y^2)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

pt: $(1,2)$

slope: $\left.\frac{dy}{dx}\right|_{(1,2)} = (1)(2)^3 = 8$

$$y - 2 = 8(x-1)$$

b) Use the tangent line equation from part *a* to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

$$f(1.1) = 8(1.1-1)+2$$
$$\approx 8(.1)+2$$
$$\approx 2.8$$

A tangent line over approximates when f is concave down and under approximates when f is concave up. Since we are given $f(x) > 0$ on $(1, 1.1)$ and we know x is positive on $(1, 1.1)$, then $\frac{d^2y}{dx^2} = y^3(1+3x^2y^2) > 0$ on $(1, 1.1)$. Since $\frac{d^2y}{dx^2} > 0$, f is concave up and the tangent line will over approximate $f(1.1)$.

c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

$$\frac{dy}{dx} = xy^3$$

y is on $(-\infty,0)\cup(0,\infty)$ →

$$\frac{dy}{y^3} = x \cdot dx$$

$$\int y^{-3}dy = \int x\,dx$$

$$-2\left[-\tfrac{1}{2}y^{-2} = \tfrac{1}{2}x^2 + C\right]$$

$$y^{-2} = -x^2 + C$$
$$\frac{1}{y^2} = -x^2 + C$$
$$y^2 = \frac{1}{-x^2+C}$$

When $x = 1$, $y = 2$

$$2^2 = \frac{1}{-(1)^2 + C}$$
$$4 = \frac{1}{-1+C}$$
$$4(-1+C) = 1$$
$$-4 + 4C = 1$$
$$4C = 5$$
$$C = 5/4$$

$$\therefore y^2 = \frac{1}{-x^2+\frac{5}{4}}$$
$$\sqrt{y^2} = \sqrt{\frac{1}{-x^2+\frac{5}{4}}}$$
$$|y| = \sqrt{\frac{1}{-x^2+\frac{5}{4}}}$$

Remember $f(1) = 2$... so y is on $(0,\infty)$ and $|y| = y$.

$$\boxed{y = \sqrt{\frac{1}{-x^2+\frac{5}{4}}}}$$