

## 4.3 Worksheet - Derivatives of Inverse Functions

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. What are the following derivatives ... these MUST be memorized forwards (now) and backwards (later).

a)  $\frac{d}{dx}[\sin^{-1} x] =$

b)  $\frac{d}{dx}[\tan^{-1} x] =$

2. Find the derivative of the following functions:

a)  $y = t \sin^{-1}(t\sqrt{2})$

b)  $f(x) = x\sqrt{1-x^2} + \cos^{-1}(x^3)$

3. A particle is moving along the  $x$ -axis so that its position at any time  $t \geq 0$  is given by  $x(t)$ . Find the velocity at the indicated value of  $t$ . Do without a calculator ... then check it with your calculator.

a)  $x(t) = \sin^{-1}\left(\frac{\sqrt{t}}{4}\right)$  when  $t = 4$ .

b)  $x(t) = \tan^{-1}(t^2)$  when  $t = 1$ .

4. Find an equation for the line tangent to the graph of  $y = \tan x$  at the point  $(\frac{\pi}{4}, 1)$ .

5. Find an equation for the line tangent to the graph of  $y = \tan^{-1} x$  at the point  $(1, \frac{\pi}{4})$ .

6. What is the relationship between the slopes of the tangent lines in questions 4 and 5? How does this help you remember the rule for finding the derivative of an inverse function?

7. [Calculator Allowed] Let  $f(x) = x^3 + 2x - 1$ .

a) Find  $f(1)$  and  $f'(1)$ .

b) How can you find  $f^{-1}(-4)$ ? Find it.

c) What is  $(f^{-1})'(-4)$ ?

d) What is  $(f^{-1})'(2)$ ?

8. [Calculator Allowed] Suppose  $f(x) = x^3 - \frac{4}{x}$

a) Find  $f(4)$  and  $f'(4)$

b) What is  $(f^{-1})'(6)$ ?

c) What is  $(f^{-1})'(-3)$ ?

9. The rule for inverse functions is that  $f(f^{-1}(x)) = x$ .

a) Take the derivative of both sides of the above expression.

b) Solve your equation from part *a* for the derivative of  $f^{-1}(x)$ .

10. The functions  $f$  and  $g$  are differentiable for all real numbers and  $g$  is strictly increasing. The table below gives values of the functions and their derivatives at selected values of  $x$ .

$x$	$g(x)$	$g'(x)$
1	2	5
2	3	1
3	4	2
4	6	7

If  $g^{-1}$  is the inverse function of  $g$ , write an equation for the line tangent to the graph of  $y = g^{-1}(x)$  at  $x = 2$ .

**7.2 Worksheet #1 - Antidifferentiation by Substitution**

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*Indefinite Integrals:* Remember the first step to evaluating any integral is to determine whether the integral is a form that should be recognized or whether  $u$ -substitution is needed.

1.  $\int (-24x^5 - 10x) dx$

2.  $\int 3x^{-1} dx$

3.  $\int 3\sec^2 x dx$

4.  $\int \frac{2x}{\sqrt{x^2+6}} dx$

5.  $\int \frac{e^x}{e^x+4} dx$

6.  $\int \frac{5}{\csc(\frac{1}{2}x)} dx$

7.  $\int \sec^2(2x) dx$

8.  $\int \frac{8}{\sqrt{1-x^2}} dx$

9.  $\int \frac{x}{2x^2+1} dx$

10.  $\int (4x^{-5} + 3^x) dx$

11.  $\int x^2 \cos(x^3) dx$

12.  $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

13.  $\int \frac{\tan^{-1} x}{1+x^2} dx$

14.  $\int \csc^2(3x+5) dx$

15.  $\int \frac{x+1}{(x^2+2x+7)^3} dx$

16.  $\int \frac{e^x}{1+2e^x} dx$

17.  $\int \sec^2(3x)e^{\tan(3x)} dx$

18.  $\int e^x(2+e^x)^{1/2} dx$

19. What is the difference between definite and indefinite integrals?

Use your prior knowledge of definite integrals to evaluate the following:

20.  $\int_1^5 \frac{3}{x} dx$

21.  $\int_{-\pi}^{\pi} \cos\left(\frac{1}{2}x\right) dx$

22.  $\int_{-1}^0 \frac{2}{6x-1} dx$

23. Let  $f$  be a differentiable function defined for all real numbers  $x$ , with the properties listed below. Find  $f(x)$ .

(i)  $f'(x) = ax^2 + bx$

(ii)  $f'(1) = 6$  and  $f''(1) = 18$

(iii)  $\int_1^2 f(x) dx = 18$

**7.2 Worksheet #2 - Antidifferentiation by Substitution**

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1. True or False:  $\int_0^{\pi/4} \tan^3(x) \sec^2(x) dx = \int_0^{\pi/4} u^3 du$

Explain your choice.

While no one is going to “force” you to do a definite integral problem using substitution a specific way, the previous problem is less likely to be missed if you get in the habit of changing the limits at the same time that you make your substitution!

2.  $\int_1^{\sqrt{2}} x \cdot 2^{-x^2} dx$

3.  $\int_e^{e^2} \frac{1}{x \ln x} dx$

4.  $\int_0^2 \sqrt{4x+1} dx$

5.  $\int_{-1}^0 \frac{2}{6x-1} dx$

6.  $\int_0^1 \frac{1}{(2x+3)^3} dx$

7.  $\int_0^{\pi} \sin\left(\frac{x}{2}\right) dx$

8. *Algebraic Techniques To Integration ...* Substitution (aka  $u$ -sub) works well when there is one part of the problem that is a derivative of the rest of the problem. When this doesn't occur, you may have to "massage" the problem to fit into a form that can be integrated from a rule or by using substitution. The more of these you do, the better you will get at recognizing which method will work. For now, use the following hints to help you get started:

a) Long Division ... You should use this when you see ...

b) Complete the Square ... You should use this when you see ...

c) Separate the Numerator ... You should use this when you see ...

d) Expand ... You should use this when you see ...

9.  $\int \frac{x^5 - 35x}{x^2 + 6} dx$

10.  $\int \frac{x^2}{x+1} dx$

11.  $\int \frac{dx}{x^2 - 4x + 4}$

12.  $\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$

~~$$13. \int \frac{3 dx}{(x+1)\sqrt{x^2+2x}}$$~~

$$14. \int \frac{x^4 + x - 4}{x^2 - 2} dx$$

$$15. \int \frac{e^x + 4}{e^x} dx$$

$$16. \int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}} dx$$

$$17. \int (x^3 - 7)^2 dx$$

$$18. \int \frac{5 - e^x}{e^{2x}} dx$$



19. [No Calculator] ... Consider a differentiable function  $f$  having domain all positive real numbers, and for which it is known that  $f'(x) = (4 - x)x^{-3}$  for  $x > 0$ .

a) Find the  $x$ -coordinate of the critical point of  $f$ . Determine whether or not the point is a relative maximum, a relative minimum, or neither for the function  $f$ . Justify your answer.

b) Find all intervals on which the graph of  $f$  is concave down. Justify your answer.

c) Given that  $f(1) = 2$ , determine the function  $f$ .

The following two integrals involve a “twist” to the normal substitution method.

After you make your normal substitution for  $u$ , you have not accounted for all of the integrand ... replace the remaining  $x$ 's by solving your substitution rule for  $x$  in terms of  $u$ .

20.  $\int (x+1)\sqrt{2-x} \, dx$

**7.2 Worksheet #3 - Antidifferentiation by Substitution**

*All work must be shown in this course for full credit. Unsupported answers may receive NO credit.*

1. Integrate each of the following:

~~a)  $\int \sin^2(x) dx$~~

b)  $\int \frac{dx}{9+4x^2}$

~~c)  $\int \sec\left(\frac{x}{7}\right) dx$~~

~~d)  $\int (8 \tan^{-1}(8t)) dt$~~

e)  $\int \frac{dx}{2+9x^2}$

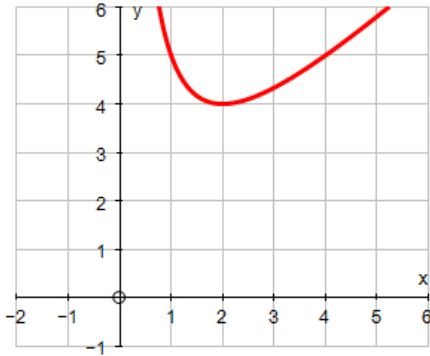
f)  $\int \frac{dx}{\sqrt{4-25x^2}}$

g)  $\int x \tan^2(x^2) dx$

~~h)  $\int \sec(6x) dx$~~

2. [No Calculator] A graph of each function is given. **Shade** each region bounded by the graphs of the equations, then find the area of that region.

a)  $y = \frac{x^2 + 4}{x}$ ,  $x = 1$ ,  $x = 4$ , and  $y = 0$



Got substitution down? This question will determine how well you truly understand EVERY aspect of the concept. Make your selection carefully.

3. If the substitution  $\sqrt{x} = \sin y$  is made in the integrand of  $\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{\sqrt{1-x}} dx$ , the resulting integral is

A  $\int_0^{\frac{1}{2}} \sin^2 y dy$

B  $2 \int_0^{\frac{1}{2}} \frac{\sin^2 y}{\cos y} dy$

C  $2 \int_0^{\frac{\pi}{4}} \sin^2 y dy$

D  $\int_0^{\frac{\pi}{4}} \sin^2 y dy$

E  $2 \int_0^{\frac{\pi}{6}} \sin^2 y dy$

4. Integrate each indefinite integral using any method possible. All Mixed UP!

a)  $\int \cos(3x)e^{\sin(3x)} dx$

b)  $\int \frac{1}{x \ln(3x)} dx$

c)  $\int \frac{\sin(3x)}{1 + \cos(3x)} dx$

d)  $\int \frac{1}{x^2 - 2x + 17} dx$

e)  $\int \frac{1}{\sqrt{1-9x^2}} dx$

f)  $\int x \csc(3x^2) \cot(3x^2) dx$

g)  $\int \frac{1 - e^{-x}}{x + e^{-x}} dx$

h)  $\int \frac{x^2 - 1}{x^2 + 1} dx$

i)  $\int \frac{x + 2}{\sqrt{4 - x^2}} dx$

5. Integrate each definite integral using any method possible. All Mixed UP!

a)  $\int_0^1 xe^{-x^2} dx$

b)  $\int_{-1}^1 \frac{1}{1+x^2} dx$

c)  $\int_0^2 (2^x + x^2) dx$

## 7.1 Worksheet - Slope Fields and Differential Equations

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. A general solution to a differential equation will have a \_\_\_\_\_ in the solution.
2. Find the general solution to the differential equations below: (need more practice? ... page 327 #2 and #4)

a)  $\frac{dy}{dx} = 5x^4 + \sec^2 x$

b)  $\frac{dy}{dx} = \sin x - e^{-x} + 8x^3$

c)  $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{x}}$

d)  $\frac{dy}{dx} = 5^x \ln(5) - \frac{1}{x^2 + 1}$

3. Find the particular solution  $y = f(x)$  using the given initial condition. How are these different than solutions in the last question? (need more practice? ... p327 #11, 12, 14, 16, 17, 20)

a)  $\frac{dy}{dx} = -\frac{1}{x^2} - \frac{3}{x^4} + 12$  and  $y = 3$  when  $x = 1$ .

b)  $\frac{dx}{dt} = \frac{1}{t} - \frac{1}{t^2} + 6$  and  $x = 0$  when  $t = 1$ .

c)  $\frac{du}{dx} = 7x^6 - 3x^2 + 5$  and  $u = 1$  when  $x = 1$ .

d)  $\frac{dv}{dt} = 4 \sec t \tan t + e^t + 6t$  and  $v = 5$  when  $t = 0$ .

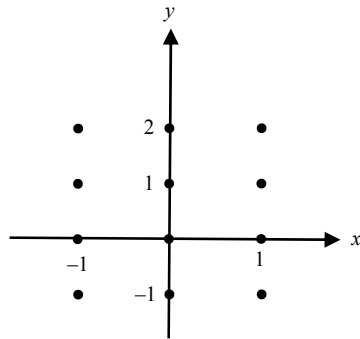
From your textbook ... page 322 ...

“An initial condition determines a particular solution by requiring that a solution curve pass through a given point. If the curve is continuous, this pins down the solution on the entire domain. If the curve is discontinuous, the initial condition only pins down the continuous *piece of the curve* that passes through the given point. In this case, the domain of the solution must be specified.”

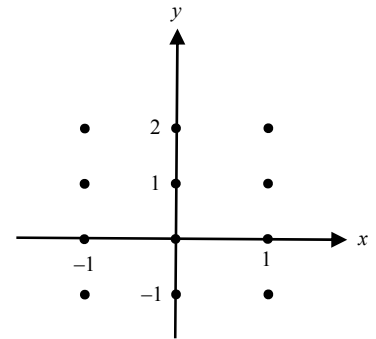
4. Which (if any) of the examples in question 3, require you to specify a domain? What are those domains?

5. Construct a slope field for each differential equation. Draw tiny segments through the twelve lattice points shown in the graph.

a)  $\frac{dy}{dx} = 2y$



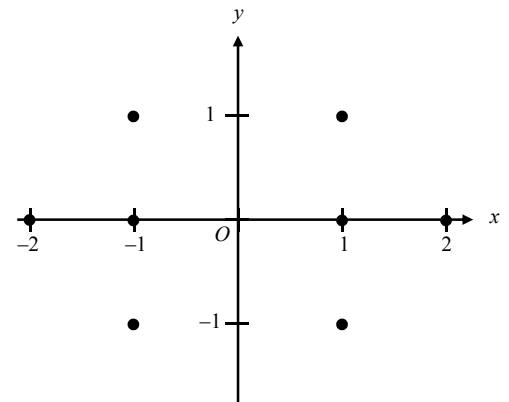
b)  $\frac{dy}{dx} = \frac{x}{2y}$



6. For each slope field above, sketch the solution curve that passes through the point (0, 1).

7. [No Calculator] Consider the differential equation  $\frac{dy}{dx} = \frac{y}{x}$ , where  $x \neq 0$ .

a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.



b) Find the particular solution  $y = f(x)$  to the differential equation with the initial condition  $f(-1) = 1$  and state its domain.

8. Use separation of variables to solve each differential equation. Indicate the domain over which the solution is valid.

a)  $\frac{dy}{dx} = \frac{x}{y}$  and  $y = 2$  when  $x = 1$ .

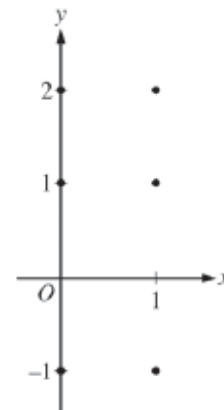
b)  $\frac{dy}{dx} = -2xy^2$  and  $y = 0.25$  when  $x = 1$ .

Even though a differential equation can be given in a problem, sometimes we are not **solving** that differential equation...try #9.

9. Consider the differential equation  $\frac{dy}{dx} = 2x - y$

a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

b) Find  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$ .



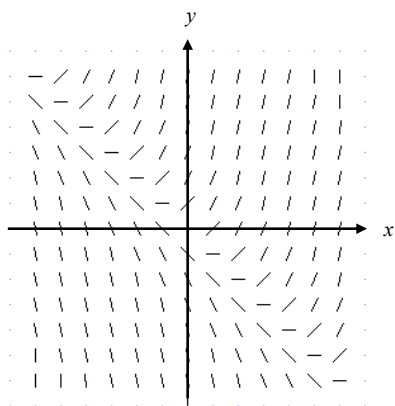
c) Let  $y = f(x)$  be the particular solution to the differential equation with the initial condition  $f(2) = 3$ . Does  $f$  have a relative minimum, a relative maximum or neither at  $x = 2$ ? Justify your answer.

d) Find the values of the constants  $m$  and  $b$  for which  $y = mx + b$  is a solution to the differential equation.

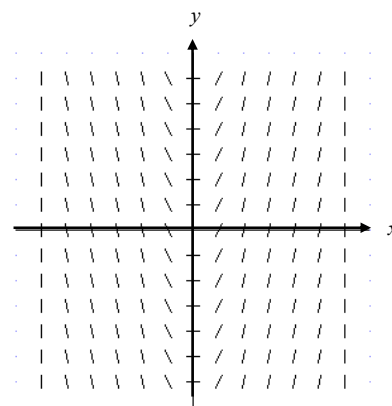


10. Solve the initial value problem  $\frac{d^2y}{dx^2} = 2 - 6x$  given that  $y(0) = 1$  and  $y'(0) = 4$ .

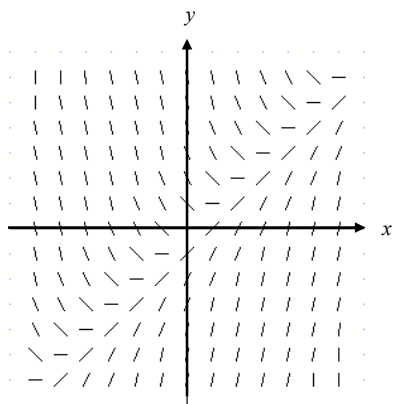
11. [MATCHING] Connect each of the six slope fields shown below to their differential equations. Explain each choice.



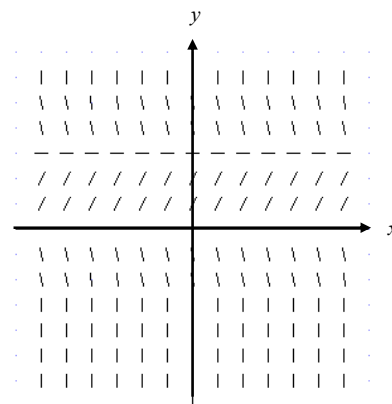
$$\frac{dy}{dx} = x - y$$



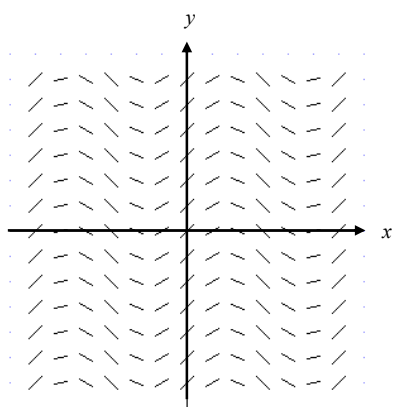
$$\frac{dy}{dx} = 2x$$



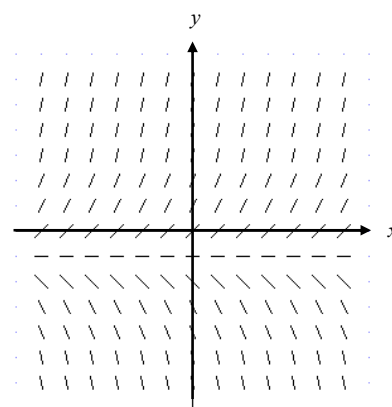
$$\frac{dy}{dx} = 1 + y$$



$$\frac{dy}{dx} = \cos x$$



$$\frac{dy}{dx} = x + y$$



$$\frac{dy}{dx} = y(3 - y)$$

## 7.4 Worksheet - Exponential Growth and Decay

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Suppose the rate of change of  $\odot$  is proportional to the amount of  $\odot$  present.

a) Write the differential equation that this statement represents.

b) Solve the differential equation from part a ... do not skip ANY steps.

2. Find the particular solution  $y = f(x)$  to each differential equation using the given initial value.

a)  $\frac{dy}{dx} = (y + 5)(x + 2)$  and  $y = 1$  when  $x = 0$ .

b)  $\frac{dy}{dx} = \frac{1}{5}(8 - y)$  and  $y = 6$  when  $x = 0$

c)  $\frac{dy}{dx} = \cos^2 y$  and  $y = 0$  when  $x = 0$ .

d)  $\frac{dy}{dx} = e^{x-y}$  and  $y = 2$  when  $x = 0$ .

3. [No Calculator] The rate of change in the population of a group of elk in the local national forest is proportional to the difference between the maximum number of elk the forest can support and the number of elk currently present. At time  $t = 0$ , when the number of elk are first counted, there are 40 elk. If  $L(t)$  is the number of elk at time  $t$  years after they are first counted, then

$$\frac{dL}{dt} = \frac{1}{4}(500 - L)$$

a) Are the elk increasing in number faster when there are 160 or when there are 360? Explain and use correct notation.

b) Find an equation for  $\frac{d^2L}{dt^2}$  in terms of  $L$ . What does  $\frac{d^2L}{dt^2}$  tell you about the graph of  $L$ ?

c) Use separation of variables to find the particular solution to  $\frac{dL}{dt} = \frac{1}{4}(500 - L)$  if  $L(0) = 40$ .

4. [No Calculator] If  $\frac{dy}{dt} = -2y$  and if  $y = 1$  when  $t = 0$ , what is the value of  $t$  for which  $y = \frac{1}{2}$ ?

- A)  $-\frac{1}{2} \ln 2$
- B)  $-\frac{1}{4}$
- C)  $\frac{1}{2} \ln 2$
- D)  $\frac{\sqrt{2}}{2}$
- E)  $\ln 2$

5. [Calculator] A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

- A) 4.2 pounds
- B) 4.6 pounds
- C) 4.8 pounds
- D) 5.6 pounds
- E) 6.5 pounds

6. [Calculator] During the zombie invasion of a small town the number of infected people is proportional to the difference between the town's population and the number of zombies currently roaming around. There are 8 zombies roaming around when they are first discovered (call this time  $t = 0$  hours). If  $Z(t)$  represents the number of zombies roaming the town at time  $t$ , then

$$\frac{dZ}{dt} = 0.05(1100 - Z)$$

a) Find a tangent line to the graph of  $Z$  when  $t = 0$ .

b) Find  $\frac{d^2Z}{dt^2}$  in terms of  $Z$ .

c) Use your tangent line from part *a* to estimate the number of zombies roaming the town 24 hours after they are first discovered ( $t = 24$ ). Is this an over approximation or an under approximation? Explain.

d) Use separation of variables to find the particular solution for  $Z(t)$  if  $Z(0) = 8$ .