AP Calculus

6.1 Estimating with Finite Sums

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

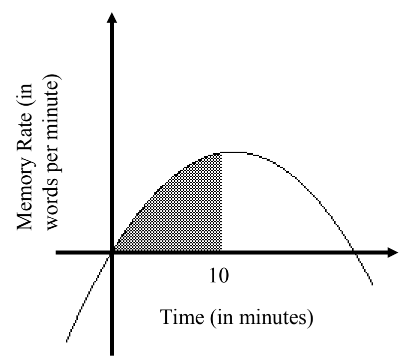
1. Suppose an oil pump is producing 800 gallons per hour for the first 5 hours of operation. For the next 4 hours, the pumps production is increased to 900 gallons per hour, and then for the next 3 hours, the production is cut to 600 gallons per hour.

a) Make a graph modeling this situation.

b) The term "area under a graph" is the area between the graph and the horizontal axis. Find the area under the graph from 0 to 5 hours. ***<u>What does this value represent</u>***?

c) Find the total area under the graph for the entire 12 hours. What does this value represent?

2. Suppose that in a memory experiment, the rate of memorizing is given by $M(t) = -0.009t^2 + 0.2t$, where $M(t)$ is the memory rate, in words per minute. The graph is shown below. Explain what the shaded area represents in the context of this problem.



3. If f is a positive, continuous function on an interval [*a*, *b*], which of the following rectangular approximation methods has a limit equal to the actual area under the curve from *a* to *b* as the number of rectangles approaches infinity?

I. LRAM II. RRAM III. MRAM

A I and II only
B III only
C I and III only
D I, II, and III
E None of these

4. The function f is continuous on the closed interval [2, 8] and has values that are given in the table below.

x	2	5	7	8
$f(x)$	10	30	40	20

Using the subintervals [2, 5], [5, 7], and [7, 8], what are the following approximations of the area under the curve? Be sure to show the correct setup for each approximation.

a) LRAM

b) RRAM

c) Trapezoid Approximation

d) Write an algebraic expression (you don't have enough information to simplify it) that would give an approximation of the area under the curve using MRAM.

5. Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the table.

a) Find an estimate using a Midpoint Sum for the total quantity of oil that has escaped in the first 8 hours using 4 intervals of equal width.

Time (h)	Leakage (gal/h)
0	50
1	70
2	97
3	136
4	190
5	265
6	369
7	516
8	720

b) Without calculating them, will LRAM or RRAM yield a "higher" estimate in this case? Why?

6. Let R be the region enclosed between the graphs of $y = 2x - x^2$ and the x-axis for $0 \le x \le 2$.

a) Sketch the region R.

b) Partition [0, 2] into 4 subintervals and find the following: (Just set it up!)

i) LRAM

ii) RRAM

iii) MRAM

iv) Trapezoidal Approximation

7. Sylvie's Old World Cheeses has found that the cost, in dollars per kilogram, of the cheese it produces is

$$c(x) = -0.012x + 6.50,$$

where x is the number of kilograms of cheese produced and $0 \le x \le 300$.

a) Draw a sketch of the cost function. Label each axes with the correct units.

b) Find the total cost of producing 200 kg of cheese. How is this represented on the graph?

8. A truck moves with positive velocity $v(t)$ from time $t = 3$ to time $t = 15$. The area under the graph of $v(t)$ between $t = 3$ and $t = 15$ gives

A the velocity of the truck at $t = 15$
B the acceleration of the truck at $t = 15$
C the position of the truck at $t = 15$
D the distance traveled by the truck from $t = 3$ to $t = 15$
E The average position of the truck in the interval $t = 3$ and $t = 15$.

9. A particle is moving along the x-axis with velocity given by $v(t) = 2t + 1$, where velocity is measured in feet/sec.

a) Draw a sketch of the velocity function for $0 \le t \le 5$.

b) What does the area under the graph of velocity represent?

c) If the object originally began at $x = 3$, where is the object located at $t = 2$? … What about $t = 5$?

10. Complete each sentence with ALWAYS, SOMETIMES, or NEVER.

a) If $f(x)$ is concave up, then LRAM will ____________ overestimate the actual area under the curve.

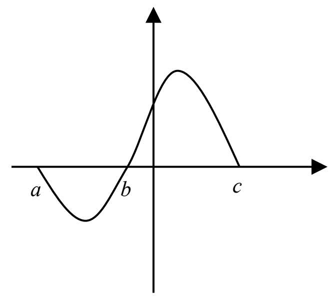
b) If $f(x)$ is decreasing, then RRAM will _____________ overestimate the actual area under the curve.

6.2 Definite Integrals

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. Graphically speaking, if $f(x)$ is always above the x-axis, what does $\int_a^b f(x)\,dx$ mean?

2. Given the graph of $f(x)$ below, answer the following questions:



   a) Is $\int_a^b f(x)\,dx$ positive, negative, or zero? Why?

   b) Is $\int_b^c f(x)\,dx$ positive, negative, or zero? Why?

   c) Is $\int_a^c f(x)\,dx$ positive, negative, or zero? Why?

3. Use your knowledge of the graph of $y = x^3$, your understanding of area, and the fact that $\int_0^1 x^3\,dx = \frac{1}{4}$ to answer the following: (Draw a sketch for each one!)

   a) $\int_{-1}^{1} x^3\,dx$ b) $\int_0^1 \left(x^3+3\right)dx$ c) $\int_0^1 \left(x^3-1\right)dx$

4. Draw a sketch and shade the "area" indicated by each integral, then use geometry to evaluate each integral.

   a) $\int_{1/2}^{3/2} (-2x+4)\,dx$ b) $\int_{-4}^{0} \sqrt{16-x^2}\,dx$ c) $\int_{-1}^{1} \left(2-|x|\right)dx$

5. If $\int_2^5 f(x)\,dx = 18$, then $\int_2^5 (f(x)+4)\,dx = $?

6. Use areas to evaluate $\int_a^b 2s\,ds$, where a and b are constants and $0 < a < b$

7. Which of the following quantities would NOT be represented by the definite integral $\int_0^8 70\,dt$?

A) The distance traveled by a train moving 70 mph for 8 minutes
B) The volume of ice cream produced by a machine making 70 gallons per hour for 8 hours
C) The length of a track left by a snail traveling at 70 cm per hour for 8 hours
D) The total sales of a company selling $70 of merchandise per hour for 8 hours
E) The amount the tide has risen 8 min after low tide if it rises at a rate of 70 mm per minute during that period

8. Express the desired quantity as a definite integral and then evaluate using geometry.

a) Find the distance traveled by a train moving at 87 mph from 8:00 AM to 11:00 AM

b) Find the output from a pump producing 25 gallons per minute during the first hour of its operation.

c) Find the calories burned by a walker burning 300 calories per hour between 6:00 PM and 7:30 PM.

d) Find the amount of water lost from a bucket leaking 0.4 liters per hour between 8:30 AM and 11:00 AM.

9. Draw a sketch for the area enclosed between the x-axis and the graph of $y = 4 - x^2$ from $x = -2$ to $x = 2$.

a) Set up a definite integral to find the area of the region.

b) Use your calculator to evaluate the integral expression you set up in part a.

All continuous functions can be integrated. But unlike derivatives, there are *some* discontinuous functions that can be integrated.

10. Consider the function $h(x) = \dfrac{x^2 - 1}{x - 1}$.

a) Is $h(x)$ is continuous on the interval [0, 3] ? If not, describe the discontinuity.

b) Sketch the region defined by $\int_0^3 h(x)\,dx$, then use geometry to evaluate the integral.

Check your answer with your calculator.

11. Consider the function $g(x) = \dfrac{|x|}{x}$.

a) Is $g(x)$ continuous on the interval [–2, 4]? If not, describe the discontinuity.

b) Sketch the region defined by $\int_{-2}^4 g(x)\,dx$, then use geometry to evaluate the integral.

Check your answer with your calculator.

12. The expression $\dfrac{1}{20}\left(\sqrt{\dfrac{1}{20}} + \sqrt{\dfrac{2}{20}} + \sqrt{\dfrac{3}{20}} + \cdots + \sqrt{\dfrac{20}{20}}\right)$ is a Riemann sum approximation for

A $\displaystyle\int_0^1 \sqrt{\frac{x}{20}}\,dx$

B $\displaystyle\int_0^1 \sqrt{x}\,dx$

C $\displaystyle\frac{1}{20}\int_0^1 \sqrt{\frac{x}{20}}\,dx$

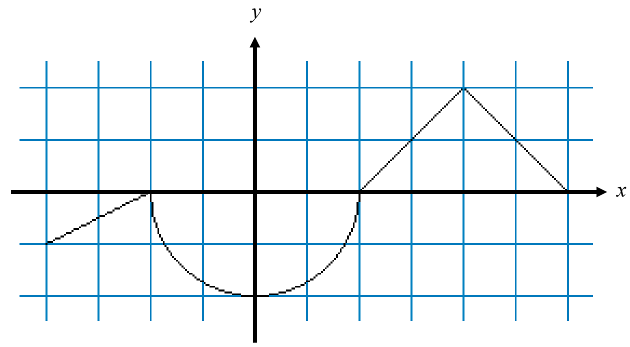
D $\displaystyle\frac{1}{20}\int_0^1 \sqrt{x}\,dx$

E $\displaystyle\frac{1}{20}\int_0^{20} \sqrt{x}\,dx$

6.3 Definite Integrals and Antiderivatives

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. The graph of f below consists of line segments and a semicircle. Evaluate each definite integral.



a) $\int_{0}^{2} f(x)\,dx$ b) $\int_{2}^{6} f(x)\,dx$

c) $\int_{-4}^{2} f(x)\,dx$ d) $\int_{4}^{0} f(x)\,dx$

e) $\int_{-4}^{6} |f(x)|\,dx$ f) $\int_{-4}^{6} [f(x)+2]\,dx$

2. Part *e* above, gives a way to find the total <u>area</u> between the x–axis and the function between $x = -4$ and $x = 6$. *Without using absolute value signs*, write an expression that can be used to find the total <u>area</u> between the x–axis and the function between $x = -4$ and $x = 6$.

3. Suppose that f and g are continuous and $\int_{1}^{2} f(x)\,dx = -4$, $\int_{1}^{5} f(x)\,dx = 6$, and $\int_{1}^{5} g(x)\,dx = 8$.

Find each of the following:

a) $\int_{2}^{2} g(x)\,dx$ b) $\int_{5}^{1} 7g(x)\,dx$ c) $\int_{1}^{2} 3f(x)\,dx$

d) $\int_{2}^{5} f(x)\,dx$ e) $\int_{1}^{5} [f(x)-g(x)]\,dx$ f) $\int_{1}^{5} [9f(x)+4]\,dx$

4. What are all the values of k for which $\int_{2}^{k} x^2\,dx = 0$?

A –2
B 0
C 2
D –2 and 2
E –2, 0, and 2

5. If $\int_3^7 f(x)\,dx = 5$ and $\int_3^7 g(x)\,dx = 3$, then all of the following must be true *except*

A $\int_3^7 f(x)g(x)\,dx = 15$

B $\int_3^7 [f(x)+g(x)]\,dx = 8$

C $\int_3^7 2f(x)\,dx = 10$

D $\int_3^7 [f(x)-g(x)]\,dx = 2$

E $\int_7^3 [g(x)-f(x)]\,dx = 2$

6. A driver average 30 mph on a 150-mile trip and then returned over the same 150 miles at the rate of 50 mph. He figured his average speed was 40 mph for the entire trip.

a) What was the total distance traveled?

b) What was his total time spent for the trip?

c) What was his average speed for the trip?

d) Explain the driver's error in reasoning.

7. A dam released 1000 m^3 of water at 10 m^3/min and then released another 1000 m^3 at 20 m^3/min. What was the average rate at which the water was released? Give reasons for your answer.

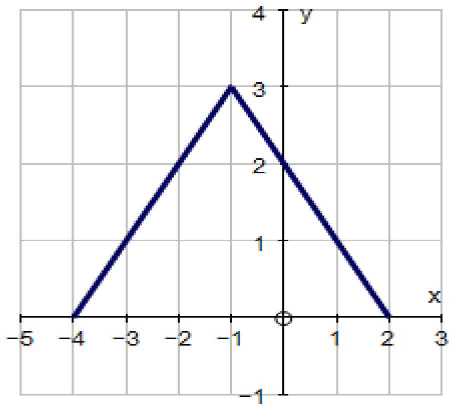
8. [Calculator] At different altitudes in Earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341 & \text{if } 0 \le x < 11.5 \\ 295 & \text{if } 11.5 \le x < 22 \end{cases}$$

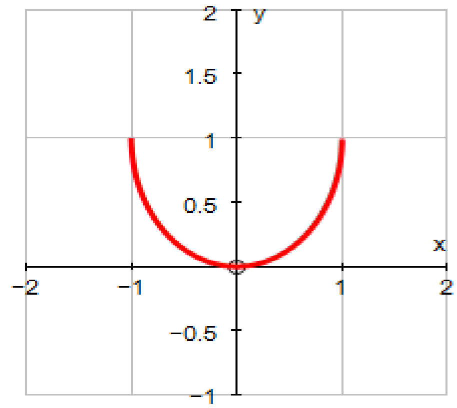
where x is measured in kilometers. What is the average speed of sound over the interval [0. 22]?

9. Find the average value of the function on the interval *without integrating*, by appealing to the geometry of the region between the graph and the x-axis.

a) $f(x)=\begin{cases} x+4 & -4\le x\le -1 \\ -x+2 & -1<x\le 2 \end{cases}$ on [–4, 2]



b) $f(x)=1-\sqrt{1-x^2}$ on [–1, 1]



10. Set up an integral to find the average value of the functions in the last question, then use your calculator to evaluate.

a)

b)

11. [Calculator] Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t)=82+4\sin\left(\frac{t}{2}\right) \quad \text{for } 0\le t\le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

a) Is traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

b) What is the average value of the traffic flow over the time interval $10\le t\le 15$? Indicate units of measure.

c) What is the average rate of change of the traffic flow over the time interval $10\le t\le 15$? Indicate units of measure

6.4 Fundamental Theorem of Calculus WS #1

All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

For questions 1 – 10, use the Fundamental Theorem of Calculus (Evaluation Part) to evaluate each definite integral. Use your memory of derivative rules and/or the chart from your notes. You should start making a list of all the rules on ONE page!

1. $\int_{1}^{4}\left(x^3+\frac{5}{\sqrt{x}}\right)dx$

2. $\int_{3}^{5}\frac{dx}{x}$

3. $\int_{1/2}^{\sqrt{3}/2}\frac{1}{\sqrt{1-x^2}}\,dx$

4. $\int_{-1}^{\sqrt{3}}\frac{1}{1+x^2}\,dx$

5. $\int_{0}^{2}5^x\,dx$

6. $\int_{-5}^{12}7x\,dx$

7. $\int_{-2}^{5}6\,dx$

8. $\int_{\pi/2}^{\pi}5\sin(x)\,dx$

9. $\int_{0}^{\pi/4}\sec^2(x)\,dx$

10. $\int_{-1}^{3}e^x\,dx$

If you would like more practice with the FTOC (Evaluation part)? … page 303 #27 – 40 (ask to borrow a book)

For questions 11 and 12, setup and evaluate an expression involving definite integrals in order to find the total AREA of the region between the curve and the *x*-axis. [No Calculator!]

11. $y = 3x^2 - 3$ on the interval $-2 \le x \le 2$

12. $y = \sqrt{x}$ on the interval $0 \le x \le 9$

For questions 13 – 16, find the average value of the function on the specified interval <u>without a calculator</u>.

13. $g(x) = 9 - 3x^2$ on the interval [0, 4]

14. $h(x) = \csc(x)\cot(x)$ on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

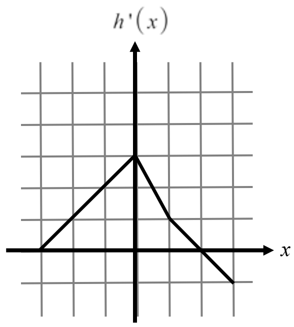
15. $y = \begin{cases} 5x & \text{if } 0 \le x \le 2 \\ 12 - x & \text{if } 2 < x \le 12 \end{cases}$

16. $f(x) = \sec^2 x$ on the interval $\left[0, \frac{\pi}{4}\right]$

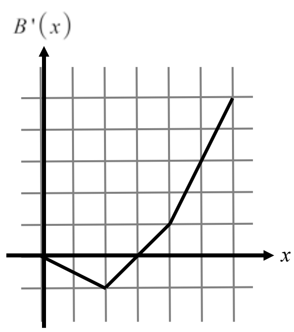
17. Including start-up costs, it costs a printer \$50 to print 24 copies of a newsletter, after which the marginal cost (in dollars per copy) at *x* copies is given by $C'(x) = \frac{2}{\sqrt{x}}$. Find the total cost of printing 2500 newsletters.

18. If you know $\int_{-7}^{9} f'(x)\,dx = 15$, and you know $f(-7) = 4$, what does $f(9) = ?$

19. The graph of $h'(x)$ is given below. If $h(-2) = 6$, what does $h(3) = ?$

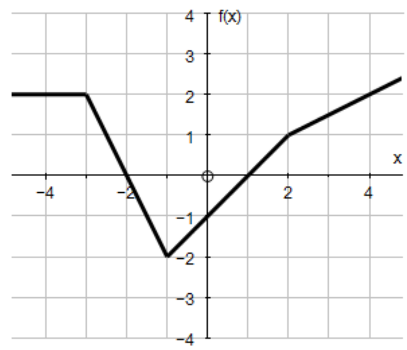


20. The graph of $B'(x)$ is given below. If you know that $B(0) = 5$, what does $B(5) = ?$



6.4 Fundamental Theorem of Calculus WS #2

1. Let $w(x)=\int_1^x f(t)\,dt$. The graph of $f(x)$ is shown below.



a) Find $w(1)$

b) Find $w(3)$

c) Find $w(-2)$

d) Find $w(-4)$

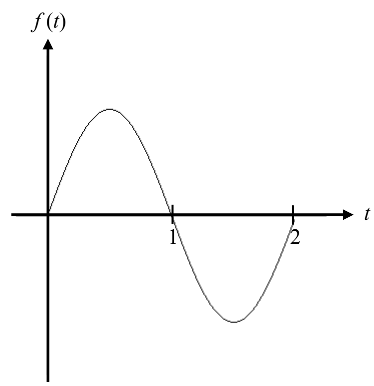
e) What is $w'(x)$?

f) Find $w'(2)$

g) $w'(-1)$

2. Let $F(x)=\int_0^x f(t)\,dt$. The graph of $f(t)$ given below has <u>odd symmetry</u> and is periodic (with <u>period = 2</u>). If you know that $\int_0^1 f(t)\,dt=\frac{4}{3}$, complete the following table:

x	F(x)
−1	
0	
1	
2	
3	



3. If a is a constant and $g(x)=\int_a^x w(t)\,dt$, what is $g'(x)$? ________________

4. If a is a constant and $g(x)=\int_x^a w(t)\,dt$, what is $g'(x)$? ________________

5. Find $\frac{d}{dx}\left[\int_{-3}^{x}\sqrt{1+e^{5t}}\,dt\right]$.

6. If $y=\int_0^x \left(t^3-t\right)^5 dt$, find y'.

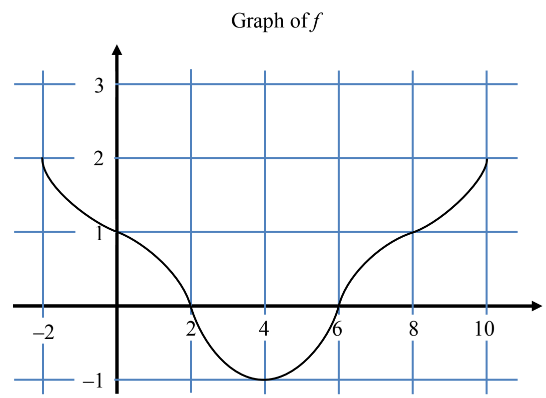
7. $k(x)=\int_{-\pi}^{x}\frac{2-\sin u}{3+\cos u}\,du$. Find $k'(x)$.

8. Find $\frac{d}{dx}\left[\int_{x}^{7}\sqrt{2p^4+p+1}\,dp\right]$

9. What is the linearization of $f(x)=\int_{\pi}^{x}\cos^3 t\,dt$ at $x=\pi$?

10. The graph of a differentiable function f on the interval [–2, 10] is shown in the figure below. The graph of f has a horizontal tangent line at $x = 4$.

Let $h(x)=9+\int_{4}^{x}f(t)\,dt$ for $-2 < x < 10$.



a) Find $h(4)$, $h'(4)$, and $h''(4)$

b) On what intervals is h increasing? Justify your answer.

c) On what intervals is h concave downward? Justify your answer.

d) Find the Trapezoidal Sum to approximate $\int_{-2}^{10}f(x)\,dx$ using 6 subintervals of length = 2.

11. If $q(x)$ and $p(x)$ are differential functions of x and $g(x)=\int_{q(x)}^{p(x)} w(t)\,dt$, what is $g'(x)$?

12. Find $\frac{d}{dx}\left[\int_{1}^{\sin x}\sqrt{1+t^3}\,dt\right]$

13. Find $\frac{d}{dx}\left[\int_{x^2}^{x^3}\cos(2t)\,dt\right]$

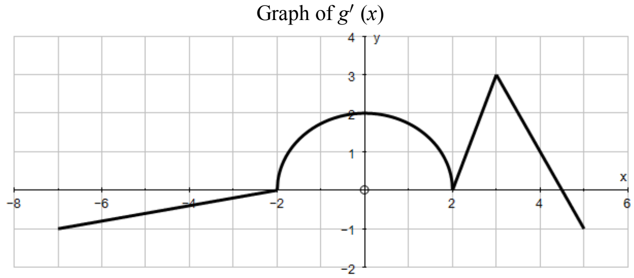
14. If $y=\int_{3x^2}^{10}\ln\left(2+u^2\right)du$, find y'.

15. Find $\frac{d}{dx}\left[\int_{\sin x}^{x^3} e^{t^2}\,dt\right]$

16. The function g is define and differentiable on the closed interval [–7, 5] and satisfies $g(0) = 5$. The graph of $g'(x)$, the derivative of g, consists of a semicircle and three line segments as shown in the figure.

a) Write an expression for $g(x)$.

Graph of $g'(x)$

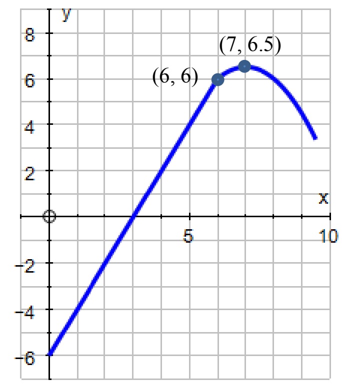


b) Use your expression to find $g(3)$ and $g(–2)$.

c) Find the x-coordinate of each point of inflection of the graph of $g(x)$ on the interval (–7, 5). Explain your reasoning.

17. Let $s(t)=\int_0^t f(x)\,dx$ be the position of a particle at time t (in seconds) as the particle moves along the x-axis. The graph of the differentiable function f is shown below. Use the graph to answer the following questions.

a) What is the particle's velocity at time $t = 4$? Justify your answer.



b) Is the acceleration of the particle at time $t = 4$ positive or negative? Justify your answer.

c) Is the particle speeding up or slowing down at time $t = 4$? Explain.

d) When does the particle pass through the origin? Explain.

e) Approximately when is the acceleration zero?

f) When is the particle moving toward the origin? Away from the origin?

g) On which side of the origin does the particle lie at time $t = 9$?