Name: ________________ **Period: ______**

Date: _______________ **AP Calc AB**

Mr. Mellina

Chapter 4: More Derivatives

Sections:

- *4.1 Chain Rule*
- *4.4 Derivatives of Exponential and Logarithmic Functions*
- *4.2 Implicit Differentiation*
- *5.6 Related Rates*

HW Sets

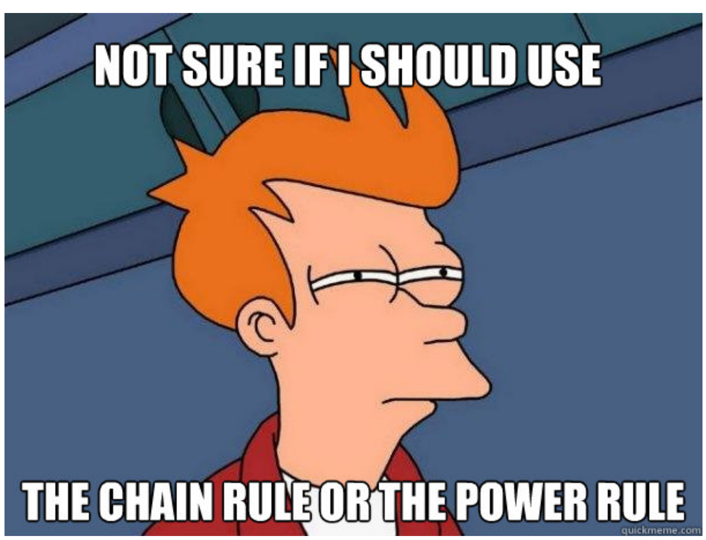
Set A (Section 4.1) Page 158, #'s 1-31 odd.

Set B (Section 4.4) Page 183, #'s 1-20, 37-40.

Set C (Section 4.2) Page 167 #'s 1-12, page 169 #'s 59 & 60.

Set D (Section 4.2) Page 167, #'s 27-30. AP Problems at end of 4.2 Section.

Set E (Section 5.6) Page 255, #'s 9, 11, 13, 17, 19.



4.1 The Chain Rule

Topics

- *Derivative of a Composite Function*
- *"Outside-Inside" Rule*
- *Repeated Use of the Chain Rule*
- *Power Chain Rule*

Warm Up!

Let $f(x) = \sin x$, $g(x) = x^2$, and $h(x) = 3x + 1$. Write a simplified expression for the composite function.

a. $f(g(x))$

b. $f\left(g\big(h(x)\big)\right)$

c. $(f \circ h)(x)$

d. $g'(h(x))$

Derivative of a Composite Function

We now know how to differentiate $\sin x$ *and* $3x + 1$*, but how do we differentiate a composite like* $\sin(3x + 1)$*? The answer is the ________ Rule, which is probably the most widely used differentiation rule in mathematics.*

The Chain Rule

Formal Definition: If f is differentiable at the point $u = g(x)$*, and g is differentiable at x, then the composite function* $(f \circ g)(x) = f(g(x))$ *is differentiable at x, and*

$$\frac{d}{dx} f\left((g(x))\right) =$$

U-Substitution:

1. *Let the "inside" function be ______.*
2. *Find ________*
3. *Substitute the inside function with u.*
4. *Differentiate with respect to ____. Don't forget to multiply by* $\frac{du}{dx}$
5. *Substitute u back in.*

Example 1: Applying the Chain Rule

Find the derivative

a. $f(x) = \sqrt{3x^2 + 17x}$

b. $h(x) = \frac{1}{x+13}$

c. $f(x) = (2x^2 + 5)^7$

d. $g(x) = \sqrt[3]{(x^2 + 1)^2}$

e. $f(x) = \left(\sqrt{x^3 - 2x^2 + 5x}\right)(x^2 + 4)$

f. $g(x) = \frac{2x}{\sqrt[3]{x^2+4}}$

g. $f(t) = \left(\frac{t-2}{2t+1}\right)^9$

Example 2: Using the Chain Rule

For each of the following, use the fact that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$ to find $f'(5)$, if possible. If it is not possible, state what additional information is required.

a. $f(x) = g(x)h(x)$

b. $f(x) = g(h(x))$

c. $f(x) = \frac{g(x)}{h(x)}$

d. $f(x) = [g(x)]^3$

Example 3: Chain Rule for Trig Functions
Find the derivative of the following

a. $y = 4x\cos x$

b. $y = \cos 4x$

c. $f(x) = \cos^2 4x$

d. $g(x) = \cos 4x^2$

Example 4: Applying the chain rule
An object moves along the x-axis so that its position at any time $t \geq 0$ is given by the following functions. Find the velocity of the object as a function of t.

a. $x(t) = \sin(t^2 + 1)$

b. $x(t) = t\cos(\pi - 4t)$

Example 5: A Three-Link Chain

Find the derivative

a. $g(t) = \tan(5 - \sin 2t)$

b. For $y = \sin^5(x)$, find $y'|_{x=\frac{\pi}{3}}$

4.4 Derivatives of Exponential & Logarithmic Functions

Topics

- *Derivatives of* e^x
- *Derivative of* a^x
- *Derivative of the* $\ln x$
- *Derivative of* $\log_a x$
- *Power Rule for Arbitrary Real Powers*

Warm Up!

Simplify the Expression using Properties of Exponents and Logarithms

a. $\ln(e^{\tan x})$

b. $\log_2(8^{x-5})$

c. $\ln(x^2-4)-\ln(x-2)$

c. $3\ln x - \ln 3x + \ln(12x^2)$

d. $e^{x\ln a}$

Example 1: Chain Rule with e

Find the Derivative

a. $f(x) = e^{x}$

b. $f(x) = e^{3x}$

c. $f(x) = xe^{x+x^2}$

d. $f(x) = e^{\sqrt{x}}$

e. $f(x) = (e^{-x} + e^{x})^3$

Example 2: Chain Rule with a^x

Find the Derivative

a. $f(x) = 3^x$

b. $f(x) = 3^{4x}$

c. $f(x) = 3^{\cot x}$

Derivative of a^x

For a >0 and $a \neq 0$, $\frac{d}{dx}(a^u) =$

Example 3: Chain Rule with $\ln x$

Find the Derivative

a. $f(x) = \ln x$

b. $f(x) = \ln(3x + 2)$

c. $g(x) = \ln(x^3 + x)$

d. $y = (\ln x)^3$

e. $y = e^{-x}\ln(x)$

f. $\log_5(x^2 - 3x)$

4.2 Implicit Differentiation

Topics

- *Implicitly Defined Functions*
- *Lenses, Tangents, and Normal Lines*
- *Derivatives of Higher Order*
- *Rational Powers of Differentiable Functions*

Warm Up!

Solve for y’ in terms of y and x.

a. $x^2y' - 2xy = 4x - y$

b. $y' \sin x - x \cos x = xy' + y$

Implicitly Defined Functions

How do we find the slope when we cannot conveniently solve the equation to find the functions. For example the function $x^3 + y^3 - 9xy = 0$ *?*

We treat y as a differentiable function of x and differentiate both sides of the equation with respect to ____, using the differentiation rules for sums, products, quotients, and the Chain Rule.

Then solve for $\frac{\quad}{\quad}$ *in terms of x and y together to obtain a formula that calculates the slope at any point (x, y) on the graph from the values of x and y. This process is called _______________ differentiation.*

Example 1: Differentiating Implicitly

Find $\frac{dy}{dx}$.

a. $y^2 = x$

b. $x^3 + y^3 = 18xy$

c. $x^2 = \frac{x-y}{x+y}$

d. $x + \sin y = xy$

Example 2: Finding Slope of a curve using Implicit Differentiation

Find the slope of the curve at the given point.

a. $x^2 + y^2 = 25,\ at\ (3, -4)$

b. $(x-1)^2 + (y-1)^2 = 25,\ at\ (3, 4)$

Example 3: Tangent and Normal to a Function

Find the lines that are tangent and normal to the curve at the given point.

a. $x^2 - \sqrt{3}xy + 2y^2 = 5$ at the point $\left(\sqrt{3}, 2\right)$

b. $x^2 \cos^2 y - \sin y = 0$ at the point $(0,\ \pi)$

Example 5: Finding a Second Derivative Implicitly

Find $\frac{d^2y}{dx^2}$

a. $2x^3 - 3y^2 = 8$

b. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$

c. $x^2 + y^2 = 1$

d. $y^2 + 2y = 2x + 1$

4.2 AP HW Problems

AP Calculus AB 2005 Test Question 5 (Form B)
Consider the curve given by $y^2 = 2 + xy$.

a. Show that $\frac{dy}{dx} = \frac{y}{2y-x}$.

b. Find all points (*x*, *y*) on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

c. Show that there are no points (*x*, *y*) on the curve where the line tangent to the curve is horizontal.

d. Let *x* and *y* be functions of time *t* that are related by the equation $y^2 = 2 + xy$. At time *t* = 5, the value of *y* is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time *t* = 5.

AP Calculus AB 2004 Test Question 4

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

a. Show that $\frac{dy}{dx} = \frac{3y-2x}{8y-3x}$.

b. Show that there is a point *P* with *x*-coordinate 3 at which the line tangent to the curve at *P* is horizontal. Find the y-coordinate of *P*.

c. Find the value of $\frac{d^2y}{dx^2}$ at the point *P* found in part (b).

5.6 Related Rates

Topics

- *Related Rate Equations*
- *Solution Strategy*
- *Simulating Related Motion*

Warm Up!

Find each of the following derivatives of $y = 2u + p - \sqrt{t}$.

a. $\frac{dy}{du}$

b. $\frac{dy}{dt}$

c. $\frac{dy}{dx}$

Example 1: Related Rate Equations

Suppose that a particle P(*x*, *y*) is moving along a curve *C* in the plane so that its coordinates *x* and *y* are differentiable function of time *t*. If *D* is the distance from the origin to *P*, then using the chain rule, we can find an equation that relates $\frac{dD}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$.

a. $D = \sqrt{x^2 + y^2}$, find $\frac{dD}{dt}$.

Any equation involving two or more variables that are differentiable functions of time t can be used to find an equation that relates their corresponding rates. These types of problems are called ___________ Rates.

When one or more values in an equation change over time, we have related rates. We use related rates when the problem asks: How fast did something ___________?

*If **h** is measured in **cm** and **t** is measured in **minutes**, then $\frac{dh}{dt}$ is measured in: ___________*

Example 2: Finding Related Rate Equations
Use the given information to write the related rate equation.

a. Assume that the radius *r* of a sphere is a differentiable equation of *t* and let *V* be the volume of the sphere. Find an equation that relates $\frac{dV}{dt}$ and $\frac{dr}{dt}$.

b. Assume that the radius *r* and height *h* of a cone are differentiable functions of *t* and let *V* be the volume of the cone. Find an equation that relates $\frac{dV}{dt}$ and $\frac{dh}{dt}$.

Strategy for Solving Related Rate Problems

1. _____________ *the problem.*
    - *In particular, identify the variable whose rate of change you* ***seek*** *and the variable (or variables) whose rate of change you* ***know****.*
2. *Develop a mathematical* _________ *of the problem.*
    - *Draw a picture (many of these problems involve geometric figures) and label the parts that are important to the problem. Be sure to distinguish constant quantities from variables that change over time. Only constant quantities can be assigned numerical values at the start.*
3. *Write an* ___________ *relating the variable whose rate of change you seek with the variable(s) whose rate of change you know.*
    - *The formula is often geometric, but it could come from a scientific application.*
4. _______________ *both sides of the equation implicitly with respect to time t. Be sure to follow all the differentiation rules. The Chain Rule will be especially critical, as you will be differentiating with respect to the parameter t.*
5. ______________ *values for any quantities that depend on time.*
    - *Notice that it is only safe to do this* ***after*** *the differentiation step. Substituting too soon, "freezes the picture" and makes changeable variables behave like constants, with zero derivatives.*
6. ______________ *the solutions.*
    - *Translate your mathematical result into the problem setting (with appropriate units) and decide whether the result makes sense.*

Example 3: Solving Related Rates
Use the given information to solve the related rate problem

a. A hot-air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 radians per minute. How fast is the balloon rising at that moment?

b. A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 miles north of the intersection and the car is 0.8 miles to the east, the police determine with radar that the distance between them and the car is increasing at 20 mpg. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

c. Water runs into a conical tank at the rate of $9\ \frac{ft^3}{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5ft. How fast is the water level rising when the water is 6ft deep?

d. Joey is perched precariously at the top of a 10 foot ladder leaning against the back wall of an apartment building (spying on an enemy of his) when it starts to slide down the wall at a rate of 4 feet per minute. Joey's accomplice, Lou, is standing on the ground 6 feet away from the wall. How fast is the base of the ladder moving when it hits Lou?

Example 4: More Practice with Related Rates

Use the given information to solve the related rate problem

a. A pebble is dropped into a pond, causing ripples in the form of concentric circles. The radius r of the outer circle is increasing at a rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

b. A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 cm/hour, while the length is increasing at a rate of 0.8 cm/hour. If the icicle is currently 4 cm in radius and 20 cm long, is the volume of the icicle increasing or decreasing, and at what rate?

c. A 15-foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of .25 feet/sec. How fast is the top of the ladder moving up the wall 12 seconds after it has been pushed?