Name: ____________________________ **Period:** ________

Date: ________________ **AP Calc BC**

Mr. Mellina/Ms. Lombardi

Chapter 8: Integration Techniques & Improper Integrals

Topics:

8.2 - Integration by Parts

8.5 -Partial Fractions

5.6 - Indeterminate Forms & L'Hôpital's Rule

8.8 - Improper Integrals

HW Sets

Section 8.2: WebAssign Chapter 8 Set A
Section 8.5: WebAssign Chapter 8 Set B
Section 5.6: WebAssign Chapter 8 Set C
Section 8.8: WebAssign Chapter 8 Set D

8.2 Integration by Parts

Topics

- *Find an antiderivative using the Tabular Method*
- *Find an antiderivative using integration by parts.*

Warm Up!

Integrate the following

a. $\int(-24x^5 - 10x)dx$

b. $\int \sec^2(3x)\, e^{\tan(3x)}\, dx$

c. $\int x^3 \sin x\, dx$

d. $\int x \ln x\, dx$

Integration by Tabular Method
Tabular integration is a special technique for integration by parts that can be applied to certain functions in the following form:

$$f(x) = g(x)h(x)$$

where one of $g(x)$ or $h(x)$ can be differentiated multiple times with ease, while the other function can be integrated multiple times with ease.

Example 1
Integrate the following.

a. $\int x^3 \sin x \, dx$

b. $\int x^4 e^{-x} \, dx$

c. $\int (x^3 + 2x) \, e^{2x} dx$

Example 2
Let *u* and v be two continuous and differentiable functions.

a. $\frac{d}{dx}[uv]$

b. Solve for $\int u dv$

Integration by by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u dv = uv - \int v du$$

This formula expresses the original integral in terms of another integral. Depending on the choices of u and dv, it may be easier to find the second integral than the original one.

Guidelines

1. *Try letting **<u>dv</u>** be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.*
2. *Try letting **<u>u</u>** be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.*

** Note that dv always includes the dx of the original integrand. **

"L-I-A-T-E" – *choose u to be the function that comes first in this list:*

L:
I:
A:
T:
E:

Example 3

Integrate the following

a. $\int x^2 \ln x \, dx$

b. $\int \sin x \ln(\cos x) dx$

c. $\int \sin^{-1} x \, dx$

d. $\int x^3\sqrt{4 - x^2} \, dx$

e. $\int x^2 \sin x \, dx$

f. $\int e^x \cos x \, dx$

g. $\int \sin^{10} x \cos x \, dx$

h. $\int e^{5x} \cos 6x \, dx$

Example 4: Independent Practice

a. $\int 3xe^{-x}\,dx$

b. $\int \frac{\ln x}{x^2}\,dx$

c. $\int x^2 \cos x\,dx$

d. $\int x \sin x \cos x\,dx$

e. $\int \cos^{-1} x\,dx$

f. $\int (\ln x)^2\,dx$

g. $\int x^3\sqrt{9-x^2}\,dx$

h. $\int e^{2x}\sin x\,dx$

i. $\int x^2\sqrt{x-1}\,dx$

j. $\int \frac{1}{x(\ln x)^3}\,dx$

k. $\int x^2\sin 4x\,dx$

8.5 Partial Fractions

Topics

- *Understand the concept of partial fraction decomposition.*
- *Use partial fraction decomposition with linear factors to integrate rational functions.*

Warm Up!

Integrate the following

a. $\int \frac{2x-1}{x^2-x-6}dx$

b. $\int \frac{4}{x-3} - \frac{1}{x+2}dx$

c. $\int \frac{3x+11}{x^2-x-6}dx$

Integration by Decomposition into Partial Fractions

1. <u>*Divide when improper*</u>*: When $N(x)/D(x)$ is an improper fraction (that is, when the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain*

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2-3 to the proper rational expression $\frac{N_1(x)}{D(x)}$.

2. <u>*Factor Denominator*</u>*: Completely factor the denominator into factors of the form*

$$(px+q)^m \quad \textit{and} \quad (ax^2+bx+c)^n$$

where ax^2+bx+c is irreducible.

3. <u>*Linear Factors*</u>*: For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.*

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

Example 1

Integrate

a. $\int \frac{1}{x^2-5x+6}\,dx$

b. $\int \frac{2}{9x^2-1}\,dx$

c. $\int \frac{3-x}{3x^2-2x-1}\,dx$

d. $\int \frac{x^3-x+3}{x^2+x-2}dx$

e. $\int \frac{x+2}{x^2+5x}dx$

f. $\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\,dx$

g. $\int \frac{\sec^2 x}{(\tan x)(\tan x + 1)}\,dx$

Example 2

Evaluate the definite integral

a. $\int_0^2 \frac{3}{4x^2+5x+1}\,dx$

b. $\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx$

5.6 Indeterminate Forms & L'Hôpital's Rule

Topics

- *Recognize limits that produce indeterminate forms.*
- *Apply L'Hôpital's Rule to evaluate a limit.*

Warm Up!

Integrate the following

a. $\lim_{x\to -1} \frac{2x^2-2}{x+1}$

b. $\lim_{x\to \infty} \frac{3x^2-1}{2x^2+1}$

L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\frac{0}{0}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $\frac{f(x)}{g(x)}$ as x approaches c produces any one of the indeterminate forms $\frac{\infty}{\infty}$, $\frac{-\infty}{\infty}$, $\frac{\infty}{-\infty}$, or $\frac{-\infty}{-\infty}$.

Example 1

The proof of L'Hôpital's Rule. Let $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = 0$

$$f(x) = \frac{g(x)}{h(x)}$$

Example 2

Evaluate the limit

a. $\lim_{x\to 0} \frac{e^{2x}-1}{x}$

b. $\lim_{x\to \infty} \frac{\ln x}{x}$

c. $\lim_{x\to -\infty} \frac{x^2}{e^{-x}}$

d. $\lim_{x\to \infty} e^{-x}\sqrt{x}$

e. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

f. $\lim_{x \to 0^+} (\sin x)^x$

g. $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

Example 3

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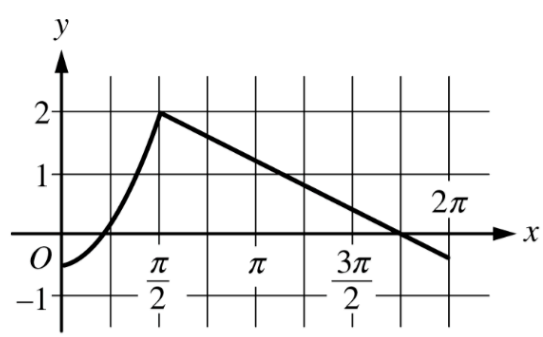
5. Let f be the function defined by $f(x) = e^x \cos x$.

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

8.8 Improper Integrals

Topics

- *Evaluate an improper integral that has an infinite limit of integration.*
- *Evaluate an improper integral that has an infinite discontinuity.*

Warm Up!

Evaluate by using your calculator.

a. $\int_1^{100} \frac{1}{x}\,dx$ b. $\int_1^{1000} \frac{1}{x}\,dx$ c. $\int_1^{1,000,000} \frac{1}{x}\,dx$ d. $\int_1^{\infty} \frac{1}{x}\,dx$

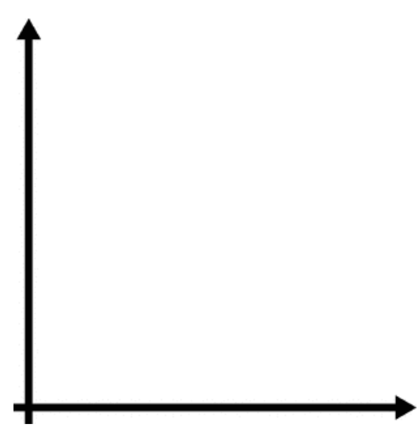
e. $\int_1^{100} \frac{1}{x^2}\,dx$ f. $\int_1^{1000} \frac{1}{x^2}\,dx$ g. $\int_1^{1,000,000} \frac{1}{x^2}\,dx$ h. $\int_1^{\infty} \frac{1}{x^2}\,dx$

Example 1

Consider the infinite region that lies under the curve $y = \frac{1}{x^2}$, above the x-axis, and to the right of the line x = 1. Shade this region.

a. Is it possible for an infinite region to have a finite area?

b. Write an integral that would describe the shaded region.



Improper Integrals

The definition of a definite integral $\int_a^b f(x)\,dx$ *requires that the interval [a, b] be finite. Furthermore, the Fundamental Theorem of Calculus requires that f be continuous on [a, b]. In this section, you will study a procedure for evaluating integrals that do not satisfy these requirements – usually because either* <u>*one or both of the limits of integration are infinite*</u> *or because* <u>*f has a finite number of infinite discontinuities*</u> *in the interval [a, b]. Integrals that possess either property are called* ***improper integrals****.*

1. *If* $\int_a^t f(x)\,dx$ *exists for every* $t \geq a$*, then* $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$
2. *If* $\int_t^b f(x)\,dx$ *exists for every* $t \leq b$*, then* $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$
3. *If f is continuous on the interval* $(-\infty, \infty)$*, then*
   $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$, *where c is a real number.*

If either yield a real, finite number, we call the integral ***convergent****. Otherwise, it is* ***divergent****.*

Example 2

Decide whether the integral is improper.

a. $\int_0^1 \frac{dx}{5x-3}$ b. $\int_1^2 \frac{dx}{x^3}$ c. $\int_0^1 \frac{2x-5}{x^2-5x+6}\,dx$ d. $\int_1^\infty \ln x^2\,dx$

e. $\int_0^2 e^{-x}\,dx$ f. $\int_0^\infty \cos x\,dx$ g. $\int_{-\infty}^\infty \frac{\sin x}{4+x^2}\,dx$ h. $\int_0^{\pi/4} \csc x\,dx$

Example 3

Decide if the following integrals converge or diverge.

a. $\int_1^\infty \frac{1}{x}\,dx$

THEOREM 8.7 A Special Type of Improper Integral

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \text{diverges}, & p \le 1 \end{cases}$$

b. $\int_{-\infty}^{0} xe^x \, dx$

c. $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$

d. $\int_1^{\infty} (1-x)e^{-x}\, dx$

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx.$$

2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx.$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

In the first two cases, the improper integral **converges** when the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

Example 4

Evaluate the definite integral.

a. $\int_2^5 \frac{1}{\sqrt{x-2}}\, dx$

b. $\int_0^3 \frac{1}{x-1}\,dx$

c. $\int_0^1 \ln x\,dx$

d. $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

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17. $\int_2^\infty \frac{1}{x^3}\,dx$

19. $\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx$

21. $\int_0^\infty e^{x/3}\,dx$

23. $\int_0^\infty x^2 e^{-x}\,dx$

25. $\int_4^\infty \frac{1}{x(\ln x)^3}\,dx$

27. $\int_{-\infty}^\infty \frac{4}{16 + x^2}\,dx$

29. $\int_0^\infty \frac{1}{e^x + e^{-x}}\,dx$

31. $\int_0^\infty \cos \pi x\,dx$

33. $\int_0^1 \frac{1}{x^2}\, dx$

35. $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\, dx$

37. $\int_0^1 x \ln x \, dx$

39. $\int_0^{\pi/2} \tan \theta \, d\theta$

41. $\int_2^4 \frac{2}{x\sqrt{x^2-4}}\, dx$

44. $\int_0^5 \frac{1}{25-x^2}\, dx$