Name: ________________ **Period: _____**

Date: ________________ **AP Calc AB**

Mr. Mellina

<u>Chapter 7: Differential Equations and Mathematical Modeling</u>

Sections:

- *4.3 Derivatives of Inverse Functions*
- *7.2 Antidifferentiation by Substitution*
- *7.1 Slope Fields and Differential Equations*
- *7.4 Exponential Growth and Decay*
- *7.5 Logistic Growth*

HW Sets

Set A (Section 4.3) Page 175, #'s 1-11 odd, 27, 28, 37, & 38.

Set B (Section 7.2) Page 342, #'s 1-6, 17-23 odd.

Set C (Section 7.2) Page 342 & 343, #'s 25, 27, 31-41 odd.

Set D (Section 7.2) Page 343, #'s 53-65 odd.

Set E (Section 7.1) Page 331, #'s 1-9 odd, 29-33 odd, 41-46.

Set F (Section 7.4) Page pg. 361, #'s 1-9 odd.

Set G (Section 7.5) Page #'s 1-21 odd.

4.3 Derivatives of Inverse Functions

Topics

- *Derivatives of Inverse Functions*
- *Derivative of Inverse Trig Functions*

Warm Up!

Verify that the following two functions are inverses of each other.

a. $f(x) = 2x^3 - 1,\ g(x) = \sqrt[3]{\frac{x+1}{2}}$

b. (2, -3) is on $f(x)$. Name a point on $f^{-1}(x)$

Example 1: Investigating Inverses

For the following use the function $f(x) = 2x + 3$. Find the following:

a. The slope of $f(x)$

b. $f^{-1}(x)$

c. The slope of $f^{-1}(x)$

d. A relationship between the slopes

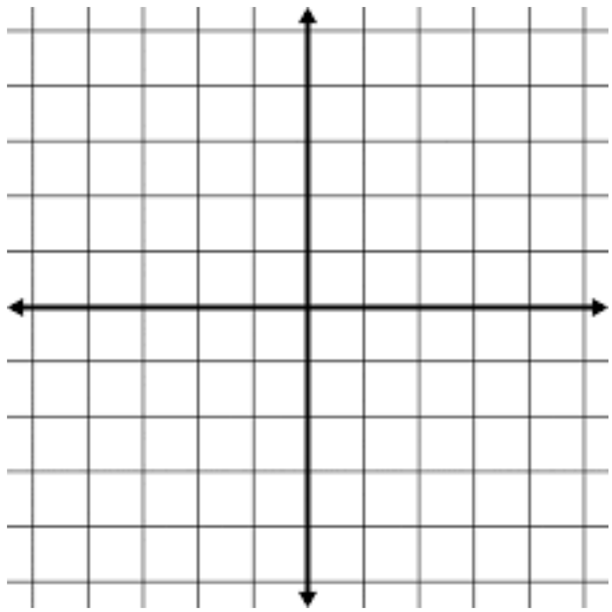
Exploration: Finding a Derivative on an Inverse Graph Geometrically

Let $f(x) = x^5 + 2x - 1$. Since the point (1, 2) is on the graph of f, it follows that the point (2, 1) is on the graph of f^{-1}. Can you find

$$\frac{df^{-1}}{dx}(2),$$

the value of $\frac{df^{-1}}{dx}$ at 2, without knowing a formula for f^{-1}?

a. Graph $f(x) = x^5 + 2x - 1$ and sketch it on the coordinate plane provided. A function must be one-to-one to have an inverse function. Is this function one-to-one?



b. Find $f'(x)$. How could this derivative help you to conclude that f has an inverse?

c. Reflect the graph of f across the line $y = x$ to obtain a graph of f^{-1}.

d. Sketch the tangent line to the graph of f^{-1} at the point (2, 1). Call it *L*.

e. Reflect the line *L* across the line $y = x$. At what point is the reflection of *L* tangent to the graph of f?

f. What is the slope of the reflection of *L*?

g. What is the slope of *L*?

h. What is $\frac{df^{-1}}{dx}(2)$

Derivatives of the Inverse Function at a point (a, b)… this implies the point (b, a) is on the original function.

To find the derivative of f^{-1} at the point (a, b) we find the _______________ of the derivative of f as the point (b, a)

$$\left.\frac{df^{-1}}{dx}\right|_a =$$

In other words, if f and g are inverses, then their derivatives at the inverse points are reciprocals.

Example 2: Derivatives of Inverses

Given $f(x)$, verify that (0, -1) is on the graph. Then find $(f^{-1})'(-1)$.

a. $f(x) = x^5 + 2x - 1$

Example 3: Derivatives of Inverse

Given $f(x)$, find $\left.\frac{df^{-1}}{dx}\right|_2$

a. $f(x) = x^3 + 2x - 1.$

Example 4: Derivatives of Inverse Trig Functions

Find $\frac{dy}{dx}$ using implicit differentiation.

a. $y = \sin^{-1} x$

b. $y = \tan^{-1} x$

Derivatives of Inverse Trigonometric Functions

$\frac{d}{dx}[\sin^{-1} x] =$

$\frac{d}{dx}[\cos^{-1} x] =$

$\frac{d}{dx}[\tan^{-1} x] =$

Example 5: Derivatives of Inverse Trig Functions

Find the derivative of the function given.

a. $f(t) = \sin^{-1}(t^2)$

b. $y = \tan^{-1}\left(\sqrt{x-1}\right)$

7.2 Integration by Substitution

Topics

- *Substitution in Indefinite Integrals*
- *Substitution in Definite Integrals*

Warm Up!

Find $\frac{dy}{dx}$.

a. $y = \int_2^x 3^t\, dt$

b. $y = \int_0^x 3^t\, dt$

c. $y = (x^3 - 2x^2 + 3)^4$

d. $y = \sin^2(4x - 5)$

e. $y = \ln \cos x$

Example 1: Antiderivatives from Derivative Rules

Complete from memory.

f	$\int f$
x^n	
$x^{-1} = \frac{1}{x}$	
e^x	
$\sin(kx)$	
$\cos(kx)$	

f	$\int f$
$\sec^2 x$	
$\csc^2 x$	
$\frac{1}{\sqrt{1-x^2}}$	
$\frac{1}{x^2+1}$	

Example 2: Remembering the Chain Rule
Use the chain rule to differentiate $f(x)$.

a. $f(x) = (3x^4 - 5)^{12}$

Example 3: Connecting Chain Rule to Anti-Derivatives
Tell whether or not each antiderivative is going to undo a chain rule.

a. $\int (x^2 - 1)^3\, 2x dx$ b. $\int 3x^2 \sqrt{x^3 + 2} dx$ c. $\int x(x^3 - 5)\, dx$

U-Substitution Method

1. *Let **u** be the "_________function" and **du** =$u'(x)dx$. Be sure to list this!!*
2. *Find $\int$*
3. *Re-substitute so expression in the terms of the original function.*

What to look for:
A function & its derivative or constant multiple of its derivative

***It is important to note, that with substitution, the goal is to substitute ALL values of the integrand with either **u** or **du**. Any "extras" must be accounted for, or substitution will NOT work!*

Example 4: Indefinite Integration with Substitution
Integrate

a. $\int x^3\sqrt{x^4 + 2} dx$

b. $\int \sin^2(3x)\cos(3x)\,dx$

c. $\int \frac{x}{x^2+2}\,dx$

d. $\int x^2\sqrt{5+2x^3}dx$

e. $\int \cot 7x \, dx$

Definite Integrals with Substitution
Indefinite Integrals needed a "+C" at the end of every antiderivative… Definite Integrals have limits.

If you change the variables, the limits still refer to the ______________ variable. How will you decide to deal with those limits? You have two choices…

1. *__________ the limits in terms of the of the original variable and integrate like you did for the indefinite integrals. Once you have returned all variables _________ to the original letter, you can plug in the upper and lower limits.*
2. *Using the rule for the change of variables, change the ___________ with the same rule, then you never need to return to the original.*

The limits must match the variable being used, or there must be some notation to indicate that the limits being used are different from the variable being used!

Example 5: Definite Integration with Substitution
Evaluate

a. Using *Method 1*. $\int_0^1 x\sqrt{1-x^2}dx$

b. Using *Method 2*. $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$

c. Using a method of your choice. $\int_{0}^{\frac{\pi}{4}} \frac{\tan^3 \theta}{\cos^2 \theta}\,d\theta$

When substitution doesn't work, you sometimes to "massage" the problem into a form that will work using some algebraic techniques. When integrating, be sure to follow the guidelines

1. *Memorized integrals*
2. *U-substitution*
3. *Change with algebra*

Example 6: Using Algebraic Techniques

Long Division – When the numerator has a degree greater than or equal to the denominator

a. $\int \frac{x^2-1}{x^2+1} dx$

Example 7: Using Algebraic Techniques

Expand – When the "inside" doesn't have a derivative on the outside, try expanding the function.

a. $\int (\sin x + \cos x)^2 \, dx$

Example 8: Using Algebraic Techniques

Complete the Square – Useful when you have a x^2 and x term in the denominator but no *x* term in the numerator.

a. $\int \frac{2\,dx}{x^2-6x+10}$

Example 9: Using Algebraic Techniques

Separate the numerator – when you have more than one term in the numerator

a. $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$

Example 10: Integration with Trig

Integrate

a. $\int \tan x \, dx$

b. $\int \cot x \, dx$

c. $\int \tan^2 x \, dx$

d. $\int \cot^2 x \, dx$

e. $\int \frac{dx}{\sqrt{4-x^2}}$

7.1 Differential Equations & Slope Fields

Topics

- *Differential Equations*
- *Slope Fields*
- *Euler's Method*

Warm Up!

Find the constant C.

a. $y = 2\sin x - 3\cos x + C$ and $y = 4$ when $x = 0$

Differential Equations

A differential equation is an equation that relates a function (or relation) with its ___________________. Just like in Algebra, when you want to solve an equation, you use an inverse operation. To "undo' a derivative we find ________________________. Recall, that a function can have many antiderivatives, all of which vary by a _______________.

The ____________ of a differential equation is the order of the highest derivative involved in the equation.

Example 1: Finding General Solutions

Find the general solution to the following differential equation. (separate and integrate)

a. $\frac{dy}{dx} = \frac{3x^2}{y}$

Finding the _______________ solution to a differential equation involves finding a unique equation that satisfies some _______________ conditions or _______________ values. In other words, instead of finding a family of functions (or relation), you are finding the function (or relation)

Example 2: Finding Particular Solutions

Find the particular solution to the following differential equation. (separate and integrate)

a. $\frac{dy}{dx} = \sin x$ if $y(0) = 2$

b. $\frac{dy}{dx} = \frac{2x}{2y - x^2 y}$ if $y(1) = 2$

A Graphical Look at Differential Equations

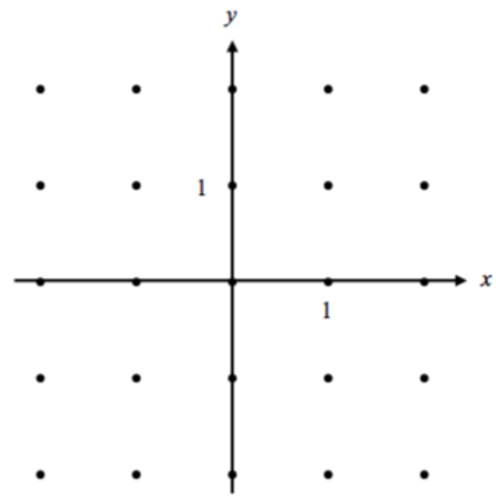
A ____________ field (or direction field) for the first order differential equation $\frac{dy}{dx} = f(x, y)$ *is a plot of short line segments with slope* $f(x, y)$ *for a lattice of points* (x, y) *in the plane.*

Example 3: Sketching Slope Fields

On the diagram provided, plot the slope field of the differential equation provided.

a. $\frac{dy}{dx} = 2y$

y	$\frac{dy}{dx}$



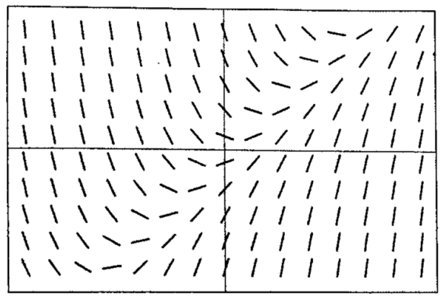
b. Suppose that you know that the point (0, -1) is on a particular solution of the differential equation above. By following slopes, draw on the diagrams above what you think the particular solution looks like. **The graph should follow the pattern of the slope field, but may go between the points rather than through them**

c. Solve the differential equation $\frac{dy}{dx} = 2y$ from the previous example by first separating the variables. Find the particular solution that contains the point (0, -1). Does your solution make sense when compared to the graph of the slope field?

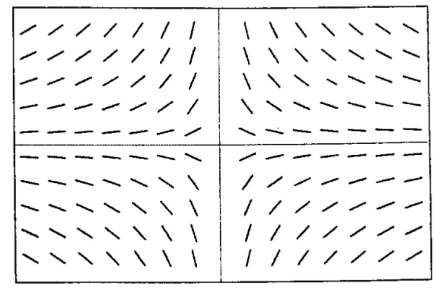
Example 4: Matching Slope Fields

Match the differential equation with the appropriate slope field. Then use the slope field to sketch the graph of the particular solution through the point (3, 2). (All slope fields are shown in the window [-6, 6] by [-4, 4].)

a. $\frac{dy}{dx} = x$

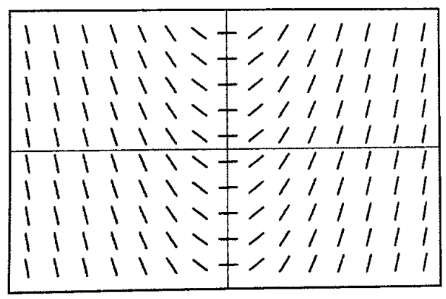


(a)

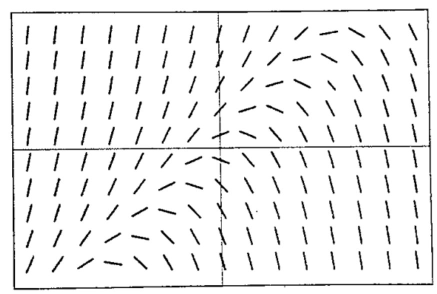


(b)

b. $\frac{dy}{dx} = y$

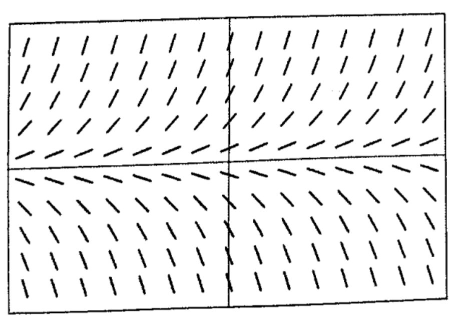


(c)

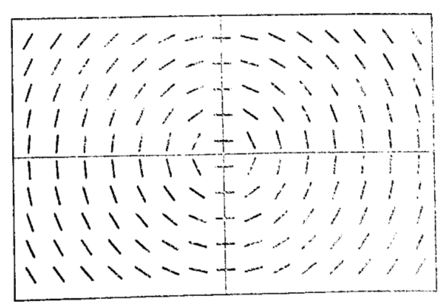


(d)

c. $\frac{dy}{dx} = x - y$



(e)



(f)

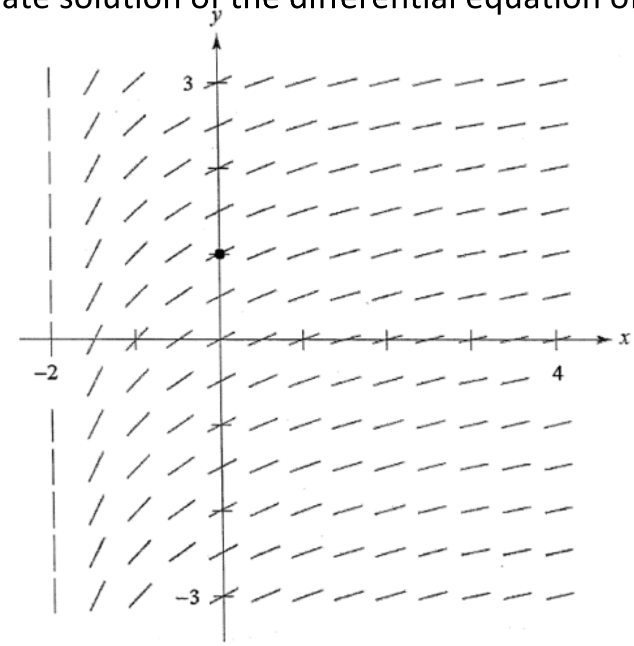
d. $\frac{dy}{dx} = y - x$

e. $\frac{dy}{dx} = -\frac{y}{x}$

f. $\frac{dy}{dx} = -\frac{x}{y}$

Example 5

Sketch an approximate solution of the differential equation of the slope field, passing through the given point.



Example 6

Which of the following is the solution to the differential equation $\frac{dy}{dx} = \frac{4x}{y}$, where $y(2) = -2$?

(A) $y = 2x$ for $x > 0$

(B) $y = 2x - 6$ for $x \neq 3$

(C) $y = -\sqrt{4x^2 - 12}$ for $x > \sqrt{3}$

(D) $y = \sqrt{4x^2 - 12}$ for $x > \sqrt{3}$

(E) $y = -\sqrt{4x^2 - 6}$ for $x > \sqrt{1.5}$

7.4 Exponential Growth and Decay

Topics

- *Exponential Growth*
- *Exponential Decay*

Warm Up!

Rewrite the equation in exponential form or logarithmic form.

a. $\ln a = b$

b. $e^c = d$

Solve the equation

c. $\ln(x + 3) = 2$

d. $2^{k+1} = 3^k$

Example 1

The rate of change of y is proportional to y. When $t = 0$, $y = 2$ and when $t = 2$, $y = 4$. What is the value of y when $t = 3$.

Example 2
Water flows continuously from a large tank at a rate proportional to the amount of water remaining in the tank. There was initially 10,000 cubic feet of water in the tank and at time t = 4 hour, 8000 cubic feet of water remained.

a. What is the value of k in the equation.

b. To the nearest cubic foot, how much water remained in the tank at time t = 8 hour?

Example 3 (No Calculator)
If a substance decomposes at a rate proportional to the amount of the substance present, and if the amount decreases from 40g to 10g in 2 hr, then the constant of proportionality is

a. -ln2 b. -0.5 c. -0.25 d. ln(0.25) e. ln(0.125)