Name: ______________________ **Period: ______**

Date: ___________ **AP Calc BC**

Mr. Mellina/Ms. Lombardi

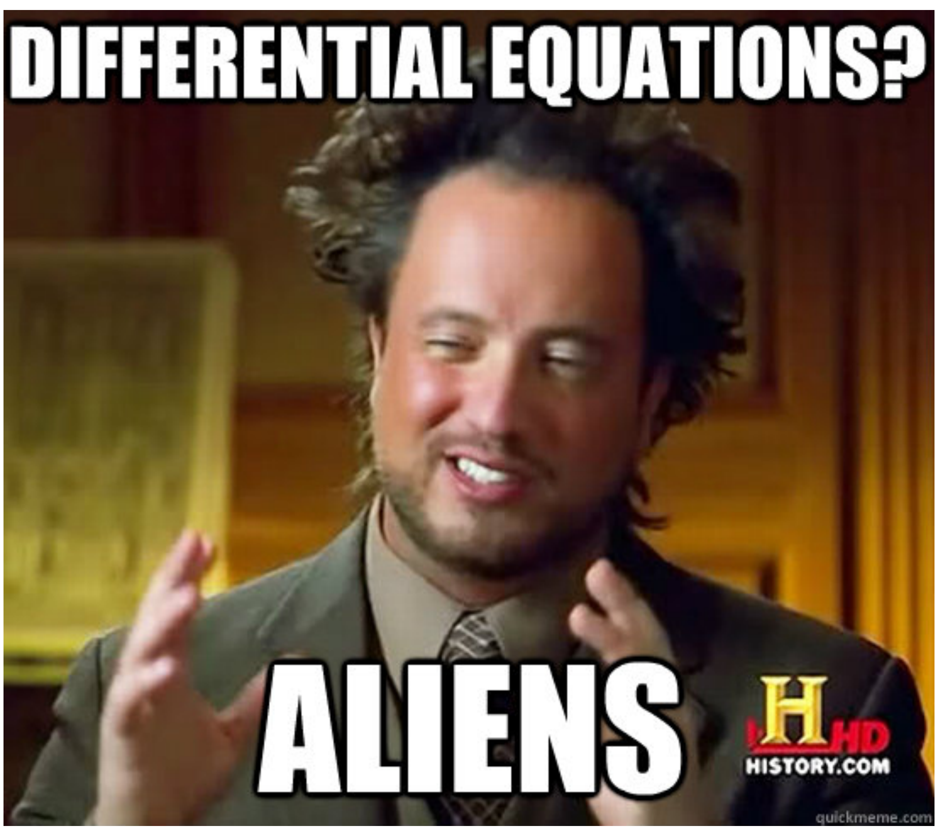
Chapter 6 – Differential Equations

Topics:

6.1 – Slope Fields and Euler's Method

6.2 – Growth & Decay

6.3 –Logistic Growth



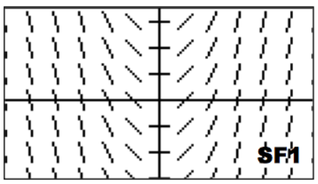
6.1 – Slope Fields and Euler's Method

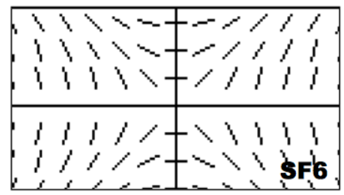
Topics

- *Use Euler's Method to approximate solutions of differential equations.*

Warm Up!

Match each slope field with its differential equation.



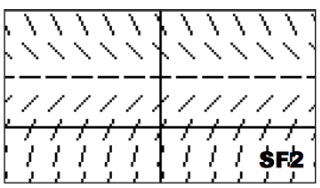


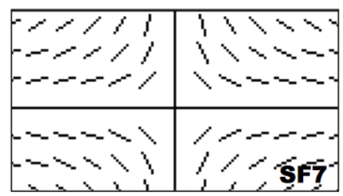
$\frac{dy}{dx} = x - y$

DE1

$\frac{dy}{dx} = \frac{x}{y}$

DE2



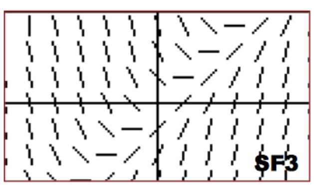


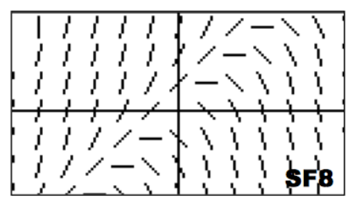
$\frac{dy}{dx} = y - x$

DE3

$\frac{dy}{dx} = -\frac{x}{y}$

DE4



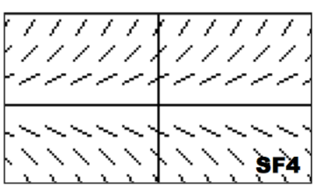


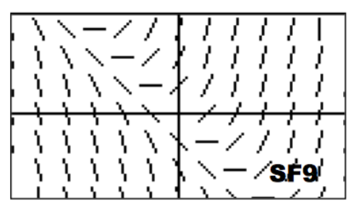
$\frac{dy}{dx} = x$

DE5

$\frac{dy}{dx} = -\frac{y}{x}$

DE6



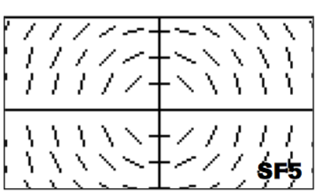


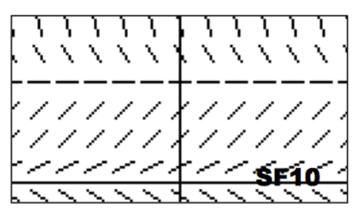
$\frac{dy}{dx} = \frac{y}{2}$

DE7

$\frac{dy}{dx} = 0.25y(4 - y)$

DE8





$\frac{dy}{dx} = 2 - y$

DE9

$\frac{dy}{dx} = x + y$

DE10

Example 1: AB Review - Finding General Solutions

Find the general solution to the exact differential equation.

a. $\frac{dy}{dx} = 5x^4 - \sec^2 x$

b. $\frac{dy}{dx} = \sin x - e^{-x} + 8x^3$

Example 2: AB Review - Finding Particular Solutions

Solve the initial value problem explicitly.

a. $\frac{dy}{dx} = 2e^x - \cos x\,,\ y = 3$ when $x = 0$

b. $\frac{dy}{dt} = \frac{1}{1+t^2} + 2^t \ln 2\,,\ y = 3$ when $t = 0$

Euler's Method

Using slope fields, we can graph the particular solution directly, by starting at the given point and piecing together little line segments to build a continuous approximation of the curve. This clever application of local linearity to graph a solution without knowing its equation is called ***Euler's Method****.*

1. *Begin at the point (x, y) specified by the initial condition. This point will beo n the graph, as required.*
2. *Use the differential equation to find the slope $\frac{dy}{dx}$ at the point.*
3. *Increase x by a small amount, Δx. Increase y by a small amount, Δy, where $\Delta y = \left(\frac{dy}{dx}\right)\Delta x$. This defines a new point $(x + \Delta x, y + \Delta y)$ that lies along the linearization. (Figure 7.7)*

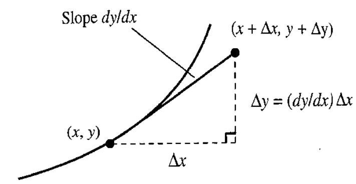


Figure 7.7 How Euler's Method moves along the linearization at the point (x, y) to define a new point $(x + \Delta x, y + \Delta y)$. The process is then repeated, starting with the new point.

4. *Using this new point, return to step 2. Repeating the process constructs the graph to the right of the initial point.*
5. *To construct the graph moving to the left from the initial point, repeat the process using negative values for Δx*

Example 3: Applying Euler's Method

Use Euler's Method with increments of $\Delta x = 0.1$ given to approximate the value of y when $x = 1.3$. Compare the approximation with the exact value.

a. $\frac{dy}{dx} = x - 1,\ y = 2$ when $x = 1$.

b. $\frac{dy}{dx} = 2x - y,\ y = 0$ when $x = 1$.

Example 4: Applying Euler's Method

Use Euler's Method with increments of $\Delta x = -0.1$ given to approximate the value of y when $x = 1.7$. Compare the approximation with the exact value.

a. $\frac{dy}{dx} = 2 - x,\ y = 1$ when $x = 2$.

b. $\frac{dy}{dx} = 1 + y,\ y = 0$ when $x = 2$.

Example 5

$x_0 = 0$	$f(x_0) = 2$
$x_1 = 2$	$f(x_1) \approx 6$
$x_2 = 4$	$f(x_2) \approx 10$

Consider the differential equation $\frac{dy}{dx} = \frac{Ax^2 + 4}{y}$, where A is a constant.

Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 2$. Euler's method, starting at $x = 0$ with a step size of 2, is used to approximate $f(4)$. Steps from this approximation are shown in the table above. What is the value of A ?

(A) $\frac{1}{2}$

(B) 2

(C) 5

(D) $\frac{13}{2}$

6.2 – Growth & Decay

Topics

- *Use separation of variables to solve a simple differential equation.*
- *Use exponential functions to model growth and decay in applied problems.*

Warm Up!

The rate of change of a variable y is proportional to the value of y. Write a differential equation that models this situation and solve for the general solution.

Growth and Decay Models

In many applications, the rate of change of a variable y is proportional to the value of y. When y is a function of time t, the proportion can be written as shown.

Rate of change of y is proportional to y.

$$\frac{dy}{dt} = ky$$

The general solution of this differential equation is given in the next theorem.

THEOREM 6.1 Exponential Growth and Decay

If y is a differentiable function of t such that $y > 0$ and $dy/dt = ky$ for some constant k, then

$$y = Ce^{kt}$$

where C is the **initial value** of y, and k is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

Example 1: Using an Exponential Growth Model

The rate of change of y is proportional to y. When $t = 0, y = 2$, and when $t = 2, y = 4$. What is the value of y when $t = 3$?

Example 2: Population Growth

An experimental population of fruit files increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Newton's Law of Cooling

The rate of change of temperature of an object is proportional to the difference between object's temperature and the temperature of the surrounding medium.

Example 3: Newton's Law of Cooling

Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°F. The object cools from 100°F to 90°F in 10 minutes. How much longer will it take for the temperature of the object to decrease to 80°F?

Extra Practice

Example 1

Write and find the general solution of the differential equation that models the verbal statement.

a. The rate of change of Q with respect to t is inversely proportional to the square of t.

b. The rate of change of P with respect to t is proportional to $25 - t$.

Example 2

Write and find the general solution of the differential equation that models the verbal statement. Evaluate the solution at the specified value of the independent.

a. The rate of change of N is proportional to N. When $t = 0$, $N = 250$, and when $t = 1, N = 400$. What is the value of N when $t = 4$?

b. The rate of change of P is proportional to P. When $t = 0, P = 5000$, and when $t = 1, P = 4750$. What is the value of P when $t = 5$?

Example 3

The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours.

a. Find the initial population

b. Write an exponential growth model for the bacteria population. Let t represent the time in hours.

c. Use the model to determine the number of bacteria after 8 hours.

d. After how many hours will the bacteria count be 25,000?

6.3 – Separation of Variables and Logistic Growth

Topics

- *Recognize and solve differential equations that can be solved by separation of variables*
- *Logistic growth as a reasonable model for population growth.*
- *Solving a logistic differential equation.*
- *Real-world applications of logistic growth.*
- *The general logistic formula.*

Warm Up!

The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$, the population is 300, and when $t = 2$, the population has increased to 500. Find the population when $t = 3$.

As the years continue to increase, does the number of coyotes approach a specific value?

Do you know what this number is called in context of the problem?

Example 1: AB Review – Separable Differential Equations

Find the general solution of the differential equation.

a. $\frac{dy}{dx} = \frac{6-x^2}{2y^3}$

b. $(2 + x)y' = 3y$

c. $yy' = -8\cos(\pi x)$

d. $y \ln x - xy' = 0$

Example 2: AB Review – Separable Differential Equations

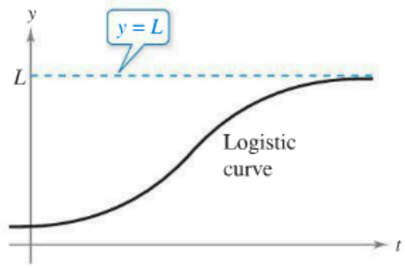
Find the particular solution of the differential equation.

a. $yy' - 2e^x = 0,\ y(0) = 6$

b. $y(1 + x^2)y' - x(1 + y^2) = 0,\ y(0) = \sqrt{3}$

How Populations Grow

We have showed that when the rate of change of a population is directly proportional to the size of the population, the population grows exponentially. This seems like a reasonable model for population growth in the short term, but populations in nature cannot sustain exponential growth for very long. Available food, habitat, and living space are just a few of the constraints that will eventually impose limits on the growth of any real-world population.



Note that as $t \to \infty$, $y \to L$.

Figure 6.16

Insight

Logistic differential equations appear only on the AP® Calculus BC Exam. On the free-response section, you may be asked to solve a logistic differential equation by separating variables. You may also be asked to find the carrying capacity and the inflection point (if any), and then explain what they represent in the context of the problem.

Logistic Differential Equation

In Section 6.2, the exponential growth model was derived from the fact that the rate of change of a variable y is proportional to the value of y. You observed that the differential equation $dy/dt = ky$ has the general solution $y = Ce^{kt}$. Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the **carrying capacity,** which is the maximum population $y(t)$ that can be sustained or supported as time t increases. A model that is often used to describe this type of growth is the **logistic differential equation**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Logistic differential equation}$$

where k and L are positive constants. A population that satisfies this equation does not grow without bound but approaches the carrying capacity L as t increases.

From the equation, you can see that if y is between 0 and the carrying capacity L, then $dy/dt > 0$, and the population increases. If y is greater than L, then $dy/dt < 0$, and the population decreases. The graph of the function y is called the *logistic curve*, as shown in Figure 6.16.

Example 3: Deriving the General Solution

Solve the logistic differential equation.

a. $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$

You can conclude that all solutions of the logistic differential equation are of the general form:

$$y = \frac{L}{1 + be^{-kt}}$$

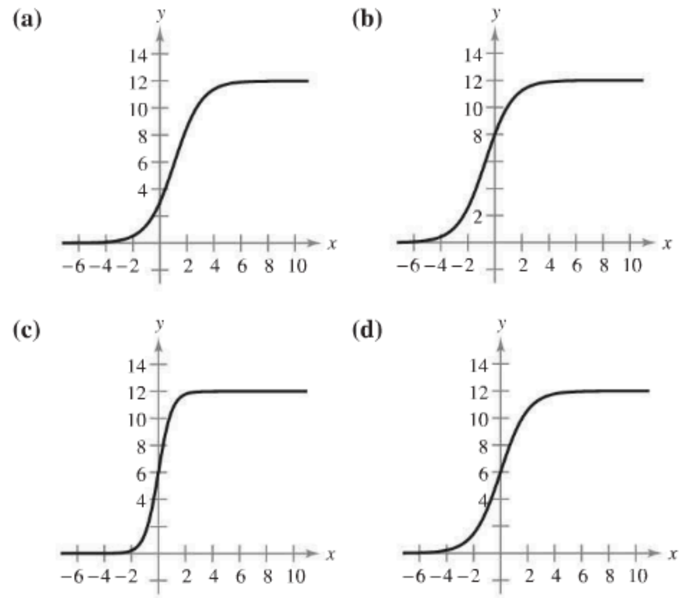
Example 4: Matching Logistic Equations with their Graphs

1. $y = \frac{12}{1+e^{-x}}$

2. $y = \frac{12}{1+3e^{-x}}$

3. $y = \frac{12}{1+\frac{1}{2}e^{-x}}$

4. $y = \frac{12}{1+e^{-2x}}$



Example 5: Using a Logistic Equation

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right), \quad 40 \le p \le 4000$$

where t is the number of years.

a. Write a model for the elk population in terms of t.

b. Graph the particular solution and use the model to estimate the elk population after 15 years.

d. Find the limit of the model as $t \to \infty$

Example 6: Weight Gain

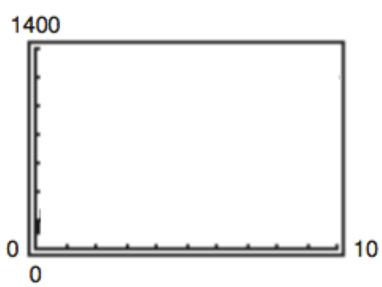
A calf that weighs 60 pounds at birth gains weight at the rate

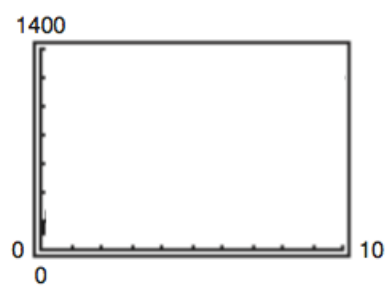
$$\frac{dw}{dt} = k(1200 - w),$$

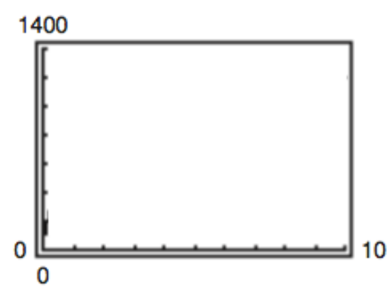
where w is the weight in pounds and t is the time in years.

a. Find the general solution of the differential equation.

b. Use a graphing utility to graph the particular solutions for $k = 0.8, 0.9$ and 1.







c. The animal is sold when its weight reaches 800 pounds. Find the time of sale for each of the models in part (b).

d. What is the maximum weight of the animal for each of the models in part (b)?

Example 7: Using a Logistic Equation

The logistic equation models the growth of a population. Use the equation to

(a) Find the value of k

(b) Find the carrying capacity

(c) Find the initial population

(d) Determine when the population will reach 50% of its carrying capacity

(e) Write a logistic equation that has the solution $P(t)$.

1. $P(t) = \frac{2100}{1+29e^{-0.75t}}$

Extra Practice

For numbers 1-4: Find the logistic equation that passes through the given point.

1. $\frac{dy}{dt} = y\left(1 - \frac{y}{36}\right),\ (0, 4)$

2. $\frac{dy}{dt} = 4.2y\left(1 - \frac{y}{21}\right),\ (0, 9)$

3. $\frac{dy}{dt} = \frac{4y}{5} - \frac{y^2}{150},\ (0, 8)$

4. $\frac{dy}{dt} = \frac{3y}{20} - \frac{y^2}{1600},\ (0, 15)$

For number 5: A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

5a. Write a logistic equation that models the population of panthers in the preserve.

5b. Find the population after 5 years.

5c. When will the population reach 100?

5d. Write a logistic differential equation that models the growth rate of the panther population.

5e. At what time is the panther population growing most rapidly? Explain.