Chapter 3: Derivatives

Sections:

- 2.4 Rates of Change & Tangent Lines
- 3.1 Derivative of a Function
- 3.2 Differentiability
- 3.3 Rules for Differentiation
- 3.4 Velocity and Other Rates of Change
- 3.5 Derivatives of Trigonometric Functions

HW Sets

Set A (Section 2.4) Page 92, #'s 1-31 odd.
Set B (Section 3.1) Pages 105-108, #'s 1-11 odd, 13-17 all, 19-23 odd, 28.
Set C (Section 3.2) Page 114 & 115, #'s 1-16 all. 31-35 odd.
Set D (Section 3.3) Pages 124, #'s 1-35 odd.
Set E (Section 3.5) Pages 146 & 147, #'s 1-35 odd.
Set F (Section 3.4) Pages 135-138, #'s 1-19, odd, 25, & 27.
2.4 Rates of Change and Tangent Lines

Topics

- Continuity at a Point
- Continuous Functions
- Algebraic Combinations
- Composites
- Intermediate Value Theorem for Continuous Functions

Warm Up!
Evaluate each limit without a calculator.

a. \( \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} \)

b. \( \lim_{h \to 0} \frac{1-2h + h^2}{h} \)

c. \( \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} \)

Average Rates of Change

We encounter average rates of change in such forms as average speed (in mph), growth rates of populations (in % per year), and average monthly rainfall (inches per month). The **Average Rate of Change** of a quantity over a period of time is the amount of change ______ divided by the ______ it takes.

In general, the average rate of change of a function over an interval is the amount of __________ divided by the ________ of the interval.

Example 1: Finding Average Rate of Change
Find the average rate of change of the function over the given interval.

a. \( f(x) = x^3 - x, [1,3] \)

b. \( f(x) = \sqrt{4x + 1}, [10,12] \)
Example 2: Considering the Slope at a Point
Let’s consider \( f(x) = x^2, [0,4] \). Approximate the slope at (1, 1)

a. Slope of Secant line through (1, 1) & (4, 16)

b. Slope of Secant line through (1, 1) & (3, 9)

c. Slope of Secant line through (1, 1) & (2, 4)

d. Slope of Secant line through (1, 1) & (1.1, 1.21)

How far can we go?
Example 3: Considering the Slope at a Point
Find the slope of the parabola \( f(x) = x^2 \) at the point P(2,4). Write the equation of the tangent line at the point provided.

Example 4: Exploring Slope and Tangent
Let \( f(x) = \frac{1}{x} \).

a. Find the slope of the curve at \( x = a \)

b. Where does the slope equal -1/4?
Example 5: Finding a Normal Line
Write an equation for the tangent & normal lines to the curve at the value provided.

a. \[ f(x) = 4 - x^2 \] at \( x = 1 \)

b. \[ f(x) = \frac{1}{x-1} \] at \( x = 2 \)
**Speed Revisited: Instantaneous Rate of Change**

The function \( y = 16t^2 \) that gave us the distance fallen by a rock in Section 2.1, was the rock’s _________ function. A body’s average speed along a coordinate axis (here, the y-axis) for a given period of time is the average rate of change of its position \( y = f(t) \). Its instantaneous ____ at any time \( t \) is the instantaneous rate of change of position with respect to time at time \( t \), or

\[
\lim_{h \to 0}
\]

**Example 6: Finding Instantaneous Rate of Change**

Find the instantaneous rate of change for the functions at the given time.

a. \( f(t) = 2t^2 - 1 \) at \( t = 2 \)

b. \( f(t) = 3t - 7 \) at \( t = 1 \)
Example 7: Estimating Derivatives with a Table

The following table lists the position \( s(t) \) of a particle at time \( t \) on the interval \( 0 < t < 4 \).

<table>
<thead>
<tr>
<th>( t ) (sec)</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s ) (ft)</td>
<td>12.5</td>
<td>26</td>
<td>36.5</td>
<td>44</td>
<td>48.5</td>
<td>50</td>
<td>48.5</td>
<td>44</td>
<td>36.5</td>
</tr>
</tbody>
</table>

a. Find the average velocity between times \( t = 0.5 \) and \( t = 1.5 \).

b. Estimate \( s'(1.5) \). Include units of measure.

c. On what interval(s) does \( s'(t) \) appear to be positive?

d. On what half second interval is the rate of change of \( s(t) \) the greatest?

A function \( T(x) \) is continuous and differentiable with values given in the table below. Use the values in the table to estimate the following.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.0</th>
<th>1.4</th>
<th>1.8</th>
<th>2.2</th>
<th>2.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T(x) )</td>
<td>1.06</td>
<td>2.2</td>
<td>3.2</td>
<td>2.8</td>
<td>3.1</td>
</tr>
</tbody>
</table>

e. \( T'(1.4) \)  
f. \( T'(2.4) \)

g. The average rate of change of \( T(x) \) between \( x = 1.4 \) and \( x = 2.2 \)

h. The instantaneous rate of change of \( T(x) \) at \( x = 1 \).

i. The equation of the tangent line to \( T(x) \) at \( x = 1 \).
3.1 Derivative of a Function

Topics

- Definition of Derivative
- Notation
- Relationships between the graphs of $f$ and $f'$$\star$
- Graphing the Derivative from Data
- One-sided Derivatives

Warm Up!
Evaluate the indicated limit algebraically

a. \[ \lim_{h \to 0} \frac{(2+h)^2 - 4}{h} \]

b. \[ \lim_{x \to 2^+} \frac{x+3}{2} \]

c. \[ \lim_{x \to 4} \frac{2x-8}{\sqrt{x}-2} \]

Definition of Derivative
In section 2.4, we defined the slope of a curve $y = f(x)$ at the point where $x = a$ to be

\[ m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

When it exists, this limit is called the ___________ of ____ at ___.

The derivative of the function $f$ with respect to the variable $x$ is the function ____ whose values at $x$ is $f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided the limit __________.

The domain of $f'$ may be smaller than the domain of $f$. If $f'(x)$ exists, we say that $f$ has a derivative (is differentiable) at $x$. A function that is differentiable at every point of its domain is a differentiable function.
### Notation

<table>
<thead>
<tr>
<th>What it sounds like</th>
<th>What it looks like</th>
<th>What it means</th>
</tr>
</thead>
<tbody>
<tr>
<td>“f prime of x”</td>
<td></td>
<td>The derivative of ___ with respect to ___</td>
</tr>
<tr>
<td>“y prime”</td>
<td></td>
<td>The derivate of ___</td>
</tr>
<tr>
<td>“dee why dee ecks”</td>
<td></td>
<td>The derivative of ___ with respect to ___</td>
</tr>
<tr>
<td>“dee eff dee ecks”</td>
<td></td>
<td>The derivative of ___ with respect to ___</td>
</tr>
<tr>
<td>“dee dee ecks uv eff uv ecks”</td>
<td></td>
<td>The derivative of ___</td>
</tr>
</tbody>
</table>

### Example 1: Applying the Definition
Use the definition of the derivative to find $f'(x)$

a. $f(x) = 2x + 3$

b. $f(x) = x^3 - 3$
**Alternate Definition of Derivative at a Point**

The derivative of the function $f$ at the point $x = a$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

---

**Example 2: Applying the Alternate Definition**

Differentiate $f(x)$

a. $f(x) = \sqrt{x}$ at the point $x = a$  

b. $f(x) = \frac{1}{x}$, at $a = 2$

---

**Example 3: Relationships Between the Graphs of $f$ and $f'$**

Consider the following function $f(x)$

a.

```
\begin{align*}
\text{Graph of } y &= f(x) \\
\text{Graph of } y &= f'(x)
\end{align*}
```

“The derivative is the __________ of the original function.”
b. Graph the derivative of the function $f$ whose graph is provided below.

Example 4: Graphing $f$ from $f'$
Sketch the graph of a function $f$ that has the following properties

a. i. $f(0) = 0$
   ii. The graph of $f'$, the derivative of $f$, is shown below.
   iii. $f$ is continuous for all $x$
b. Sketch the graph of a continuous function $f$ with $f(0) = 1$ and $f'(x) = \begin{cases} 2, & x < 2 \\ -1, & x > 2 \end{cases}$

Example 5: Comparing $f'$ with $f$
Suppose the graph below is the graph of the derivative of $h$.

a. What is the value of $h'(0)$? What does this tell us about $h(x)$?

b. Using the graph of $h'(x)$, how can we determine when the graph of $h(x)$ is going up? How about going down?

c. The graph of $h'(x)$ crosses the x-axis at $x = 2$ and $x = -2$. Describe the behavior of the graph of $h(x)$ at these points.
One-Sided Derivatives
A function \( y = f(x) \) is differentiable on a \( [a, b] \) interval if it has a derivative at every interior point of the interval, and if the following limits exist at the endpoints:

\[
\lim_{{h \to 0^+}} \frac{f(a+h) - f(a)}{h} \quad [\text{the right-hand derivative at } a]
\]

\[
\lim_{{h \to 0^-}} \frac{f(a+h) - f(a)}{h} \quad [\text{the left-hand derivative at } a]
\]

Example 6: One-Sided Derivatives Can Differ at a Point
Show that the following function has left-hand and right-hand derivatives at the given \( x \) value, but no derivative there.

a. \( y = \begin{cases} x^2, & x \leq 0 \\ 2x, & x > 0 \end{cases} \) at \( x = 0 \)

b. \( f(x) = \begin{cases} x^2 + x, & x \leq 1 \\ 3x - 2, & x > 1 \end{cases} \) at \( x = 1 \)
### 3.2 Differentiability

**Topics**
- How $f'(a)$ Might Fail to Exist
- Differentiability Implies Local Linearity
- Numerical Derivatives on a Calculator
- Differentiability Implies Continuity
- Intermediate Value Theorem for Derivatives

**Warm Up!**

Tell whether the limit could be used to define $f'(a)$ (assuming that $f$ is differentiable at $a$)

\[
\begin{align*}
\text{a. } & \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
\text{b. } & \lim_{h \to 0} \frac{f(a+h) - f(h)}{h} \\
\text{c. } & \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \\
\text{d. } & \lim_{x \to a} \frac{f(a) - f(x)}{a-x} \\
\text{e. } & \lim_{h \to 0} \frac{f(a+h) + f(a-h)}{h}
\end{align*}
\]

**How $f'(a)$ Might Fail to Exist**

A function will not have a derivative at a point $P(a, f(a))$ where the slopes of the secant lines, $\frac{f(x) - f(a)}{x-a}$, fail to approach a ________ as $x$ approaches $a$.

**Example 1: When $f'(a)$ fails to exist graphically**

A function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has:

a. A __________, where the one-sided derivatives differ

ex:
\[f(x) = |x|\]

b. A __________, where the slopes of the secant lines approach $\infty$ from one side and $-\infty$ from the other

ex:
\[f(x) = x^3\]
c. A ______________ _________, where the slopes of the secant lines approach either $\infty$ or $-\infty$ from both sides.

d. A ________________, which will cause one or both of the one-sided derivatives to be Non-existent.

ex:
\[ f(x) = \sqrt[3]{x} \]

ex:
\[ f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \]

Example 2: Finding Where a Function is Not Differentiable

Find all points in the domain of \( f(x) \) where \( f \) is not differentiable.

a. \( f(x) = |x - 2| + 3 \)  

b. \( f(x) = \begin{cases} x, & x < 1 \\ \frac{1}{x}, & x \geq 1 \end{cases} \)

Most of the functions we encounter in calculus are differentiable wherever they are defined. They will _____ have corners, cusps, vertical tangent lines, or points of discontinuity within their domains. Their graphs will be ____________ and ____________, with a well-defined slope at each point.

Polynomials are differentiable, as are rational functions, trig functions, exponential functions, and logarithmic functions. Composites of differentiable functions are differentiable, and so are sums, products, integer powers, and quotients of differentiable functions, where defined.
Example 3: Using the Calculator to Evaluate Derivatives
Use your Calculator to find the derivative at the given points.

a. \( y = x^3 \), find \( \frac{dy}{dx} \) at \( x = 2 \).

b. \( f(x) = 5x^2 + 4x \), find \( f'(x) \) at \( x = 1 \).

**WARNING**
The calculator may return an incorrect value if you evaluate a derivative at a point where the function is not differentiable.

Examples:
\[ d(1/x, x) \big|_{x=0} \text{ returns } -\infty \]
\[ d(abs(x), x) \big|_{x=0} \text{ returns } \pm 1 \]

Example 4: Using the Calculator
Graph the following functions, use the value and \( \frac{dy}{dx} \) feature to fill in the table. Then guess what you think the derivative is.

a. \( f(x) = e^x \)

\[
\begin{array}{|c|c|c|}
\hline
x & f(x) & f'(x) \\
\hline
-1 & & \\
0 & & \\
1 & & \\
2 & & \\
3 & & \\
\hline
\end{array}
\]

\[ \frac{d}{dx} (e^x) = \]

b. \( f(x) = \ln x \)

\[
\begin{array}{|c|c|c|}
\hline
x & f(x) & f'(x) \\
\hline
1 & & \\
2 & & \\
3 & & \\
4 & & \\
5 & & \\
\hline
\end{array}
\]

\[ \frac{d}{dx} (\ln x) = \]
**Differentiability Implies Continuity**

*We began this section with a look at the typical ways that a function could fail to have a derivative at a point. As one example, we indicated graphically that a discontinuity in the graph of \( f \) would cause one or both of the one-sided derivatives to be nonexistent.*

**Theorem 1: Differentiability Implies Continuity**

*If \( f \) has a derivative at \( x = a \), then \( f \) is _______________ at \( x = a \). (the converse is not true)*

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**Example 5: Differentiability implies Continuity**

If \( f \) is a function such that \( \lim_{x \to -3} \frac{f(x) - f(-3)}{x + 3} = 2 \), which of the following must be true?

a. The limit of \( f(x) \) as \( x \) approaches -3 does not exist.

b. \( f \) is not defined at \( x = -3 \).

c. The derivative of \( f \) at \( x = -3 \) is 2.

d. \( f \) is continuous at \( x = 2 \).

e. \( f(-3) = 2 \).

---

**Example 6: Sketching Graphs**

Sketch a graph of a function \( f(x) \) that meets the following criteria for \( f'(x) \).

a. \( f' \) is negative and increasing for all \( x \)

b. \( f' \) is positive and increasing for all \( x \)

c. \( f' \) is negative and decreasing for all \( x \)

d. \( f' \) is positive and decreasing for all \( x \)
3.3 Rules for Differentiation

Topics

- Positive Integer Powers, Multiples, Sums, and Differences
- Products and Quotients
- Negative Integer Powers of $x$
- Second and Higher Order Derivatives

Warm Up!

Write the expression as a sum of powers of $x$.

a. $(x^2 - 2)(x^{-1} + 1)$  
   b. $\left(\frac{x}{x^2+1}\right)^{-1}$  
   c. $3x^2 - \frac{2}{x} + \frac{5}{x^2}$

d. $\frac{3x^{4}-2x^{3}+4}{2x^{2}}$  
   e. $(x^{-1} + 2)(x^{-2} + 1)$  
   f. $\frac{x^{-1}+x^{-2}}{x^{-3}}$

g. For the given function below, graph $\frac{dy}{dx}$  
   h. Use the alternate form of a derivative to find $y'$ for $y = 5$. 

![Graph of a function](image)
**Example 1: Differentiating a Polynomial**

Find \( \frac{dp}{dt} \)

a. \( p = t^3 + 6t^2 - \frac{5}{3}t + 16 \)

b. \( p = \frac{t^3}{3} + \frac{t^2}{2} + 4t - \frac{1}{t} + 1 \)

**Example 2: Using Derivatives**

Find the following.

a. Find the slope at \( x = 2 \) of \( y = x^4 - 4x^2 + 1 \)

b. At what \( x \)-value(s) are there any horizontal tangents of \( y = x^4 - 2x^2 + 2 \)
Example 3: Using the Product and Quotient Rule
Find $f'(x)$ and simplify.

a. $f(x) = (x^2 + 1)(x^3 + 3)$

b. $f(x) = (3 + 2\sqrt{x})(5x^3 - 7)$
c. \[ f(x) = \frac{x^2 - 1}{x^2 + 1} \]

d. \[ f(x) = \frac{1 + \ln x}{x^2 - \ln x} \]
Example 4: Working with Numerical Values
Let \( y = u \cdot v \) be the product of the functions \( u \) and \( v \). Find \( y'(2) \) if
a. \( u(2) = 3, \ u'(2) = -4, \ v(2) = 1, \) and \( v'(2) = 2 \)

Example 5: Using the Power Rule
Let \( y = \frac{x^2 + 3}{2x} \). Find the following
a. \( y' \) using the quotient rule
b. \( y' \) by making the fraction a sum of two powers of \( x \).

C. Find an equation for the line tangent to \( y \) at the point \((1, 2)\)
**Second and Higher Order Derivatives**

The derivative \( y' = \frac{dy}{dx} \) is called the ______ derivative of \( y \) with respect to \( x \). The first derivative may itself be a differentiable function of \( x \). If so, its derivative

\[
y'' = \frac{dy'}{dx} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2}
\]

is called the ______ derivative of \( y \) with respect to \( x \). If \( y'' \) is differentiable its derivative is called the ______ derivative of \( y \) with respect to \( x \). The names continue as you might expect they would, except that the multiple-prime notation begins to lose its usefulness after about 3 primes. We use

\[
y^{(n)} = \frac{d}{dx} y^{(n-1)}
\]

to denote the ____ derivative.

---

**Example 6: Finding Higher Order Derivatives**

Find the first four derivatives of the function provided.

a. \( y = x^3 - 5x^2 + 2 \)

b. \( y = \frac{x+1}{x} \)
3.5 Derivatives of Trigonometric Functions

Topics

- Derivative of the Sine Function
- Derivative of the Cosine Function
- Simple Harmonic Motion
- Jerk
- Derivatives of the Other Basic Trig Functions

Warm Up!
Evaluate the limit without a calculator.

\[
\begin{align*}
\text{a. } & \lim_{x \to 0} \frac{5 \sin 3x}{4x} \\
\text{b. } & \lim_{x \to 0} \frac{\cos x - 1}{x}
\end{align*}
\]

Example 1: Derivative of Sine
Consider the following graph of Sine. Fill in the chart with the slopes at the given values of \( \theta \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>-( \pi )</td>
<td></td>
</tr>
<tr>
<td>-( \frac{\pi}{2} )</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td></td>
</tr>
<tr>
<td>( \pi )</td>
<td></td>
</tr>
</tbody>
</table>

Derivative of Sine

\[
\frac{d}{dx} \sin x =
\]
Example 2: Derivative of Cosine
Consider the following graph of Cosine. Fill in the chart with the slopes at the given values of $\theta$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\pi$</td>
<td></td>
</tr>
<tr>
<td>$-\frac{\pi}{2}$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td></td>
</tr>
<tr>
<td>$\pi$</td>
<td></td>
</tr>
</tbody>
</table>

Example 3: Revisiting the Differentiation Rules
Find the derivative

a. $y = x^2 \sin x$

b. $u = \frac{\cos x}{1-\sin x}$
Example 4: Derivative of Tangent
Find the derivative of $\tan x$ using the quotient rule.

a. $\frac{d}{dx} [\tan x]$ 

Example 5: A Trigonometric Derivative
Find the derivative of each function.

a. $f(t) = \sqrt{t} + 4 \sec t$ 

b. $h(\theta) = 5 \sec \theta + \tan \theta$

c. $h(s) = \frac{1}{s} - 10 \csc s$

d. $y = x \cot x$
3.4 Velocity and Other Rates of Change

Topics
- Instantaneous Rates of Change
- Motion Along a Line
- Sensitivity to Change
- Derivatives in Economics

Warm Up!
Answer the questions about the graph of the quadratic function \( f(x) = -16x^2 + 160x - 256 \) by analyzing the equation algebraically. (No calculator)

a. Does the graph open upward or downward?
b. What is the y-intercept?
c. What are the x-intercepts?
d. What is the range of the function?
e. For what x-value does \( \frac{dy}{dx} = 100 \)?
f. Find \( \frac{d^2y}{dx^2} \) at \( x = 7 \)

Example 1: Average vs. Instantaneous Velocity
Consider a graph of displacement (distance traveled) vs. time

Average Velocity can be found by taking:
\[
\frac{\text{change in distance}}{\text{change in time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}
\]

Instantaneous Rate of Change
The speedometer in your car does not measure average velocity, but instantaneous velocity. The velocity at one ___________ in _______.

\[
V(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t) - f(t_0)}{\Delta t}
\]

It is conventional to use the word instantaneous even when \( x \) does not represent time. The word, however, is frequently omitted in practice. When we say “rate of change,” we mean instantaneous rate of change.

Velocity is the ________ derivative of position.
Example 2: PVT (Calculator Allowed)
An object was dropped off a cliff. The object’s position is given by \( s(t) = -16t^2 + 16t + 320 \), where \( s \) is measured in feet and \( t \) is measured in seconds.

\( a. \) What is the objects displacement from \( t = 1 \) to \( t = 2 \) seconds?

\( b. \) When will the object hit the ground?

\( c. \) What is the object’s velocity at impact?

\( d. \) What is the object’s speed at impact?

\( e. \) Find the object’s acceleration as a function of time.
Example 3: PVT
Suppose the graph provided shows the velocity of a particle moving along the x-axis. Justify each response.

a. Which way does the particle move first?

b. When does the particle stop?

c. When does the particle change direction?

d. When is the particle moving left? ... moving right?

e. Graph the particle’s acceleration for $0 < t < 10$.

f. When is the particle speeding up?

g. When is the particle slowing down?

h. Graph the particle’s speed for $0 < t < 10$

i. When is the particle moving the fastest?

j. When is the particle moving at a constant speed?
Example 4: Using Derivatives
A dynamic blast propels a heavy rock straight up with a launch velocity of 160 feet per second (about 190 mph). It reaches a height of $s(t) = 160t - 16t^2$ ft after $t$ seconds.

a. How high does the rock go?

b. What is the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

c. What is the acceleration of the rock at any time $t$ during its flight (after the blast)?

d. When does the rock hit the ground?
Example 5: Studying Particle Motion
A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^2 - 4t + 3$, where $s$ is measured in meters and $t$ is measured in seconds.

a. Find the displacement of the particle during the first 2 seconds.

b. Find the average velocity of the particle during the first 4 seconds.

c. Find the instantaneous velocity of the particle when $t = 4$.

d. Find the acceleration of the particle when $t = 4$.

e. Describe the motion of the particle. At what values of $t$ does the particle change directions?

f. Graph $s(t)$ and its derivative.
It is important to understand the relationship between a position graph, velocity and acceleration: