Chapter 8: Applications of Definite Integrals

Sections:

- 8.1 Integral as Net Change
- 8.2 Areas in the Plane
- 8.3 Volumes

HW Sets

Set A (Section 8.1) Pages 390 & 391, #’s 1, 3, 7, 11-16, 19, 21
Set B (Section 8.1) Page 390 & 392, #’s 2 & 6 (no calc), 31-36 (calc)
Set C (Section 8.2) Pages 399-401, #’s 1 & 2 (calc), 3-6 (no calc), 52, 53, 55 (no calc), and 54 (calc)
Set D (Section 8.3) Page 400, #’s 9, 10, 13, 15, 36, 38 (no calc)
Set E (Section 8.4) Page 437, #’s 53-55, Page 412 #39
Set F (Section 8.4) Page 411 #’s 7-15 (Disk and Washer Method)
Set G (Section 8.4) Page 411 #’s 27, 29, 31 (Disk and Washer Method)
8.1 Integrals as Net Change

Topics
- Linear Motion Revisited
- General Strategy
- Consumption over time
- Net change from data

Warm Up!
Find all values of $x$ (if any) at which the function changes sign on the given interval. Sketch a number line graph of the interval, and indicate the sign of the function on each subinterval.

a. $\sin 2x$ on $[-3, 2]$ 

\[ \sin 2x = \sin^{-1}(0) \]
\[ 2x = -\pi, 0, \pi \]
\[ x = -\frac{\pi}{2}, 0, \frac{\pi}{2} \]

b. $x^2 - 3x + 2$ on $[-2, 4]$

\[ (x-2)(x-1) = 0 \]
\[ x = 2, x = 1 \]

Example 1
Oil is leaking from a storage tank at a rate of $r(t) = 4000e^{-3t}$ liters/day.

a. How much oil will leak in the first 5 days?

\[ \int_{0}^{5} r(t) \, dt \approx 10358.265 \text{ liters} \]

b. How much oil will leak in the next 5 days?

\[ \int_{5}^{10} r(t) \, dt \approx 2131.241 \text{ liters} \]

c. How much oil will leak in the first 50 days?

\[ \int_{0}^{50} r(t) \, dt \approx 13333.329 \text{ liters} \]
Example 2

Velocity is not the only rate in which you can integrate to get a total. We can also use integrals to represent “consumption” over a certain time period. In fact, if you were given a function that gave the number of tickets per hour that the police wrote each day, and you wanted to find the total number of tickets in a 24-hour period, you could integrate.

a. Using the above scenario, suppose the rate of ticket writing is given by \( N(h) \). Write an integral to find the total number of tickets written in 2 days.

\[
\int_0^{48} N(h) \, dh \Rightarrow \frac{\text{\# of tickets}}{\text{hr}} \cdot \text{hr} = \text{\# of tickets}
\]
Example 3
The tide removes sand from Sandy Point Beach at a rate modeled by the function $R$ given below. A pumping station adds sand to the beach at a rate modeled by the function $S$ given below. Both $R(t)$ and $S(t)$ have units of cubic yards per hour and $t$ is measured in hours for $0 \leq t \leq 6$. At time $t = 0$, the beach contains 2,500 cubic yards of sand.

$$R(t) = 2 + 5 \sin \left(\frac{4\pi t}{25}\right) \quad S(t) = \frac{15t}{1+3t}$$

a. How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

\[
\int_0^6 R(t) \, dt = 31.816 \text{ yd}^3 \text{ of sand removed in 6 hrs}
\]

b. Write an expression for $Y(t)$, the total number of cubic yards of sand on the beach at time $t$.

$$y(t) = \text{original amount} + \frac{\text{change in amount of sand after} \quad \text{change in amount of sand after}}{t \text{ hours}}$$

$$y(t) = 2500 + \int_0^t [S(x) - R(x)] \, dx$$

c. Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

\[
y'(4) = S(4) - R(4) = -1.909 \text{ yd}^3/\text{hr}
\]

want $y'(4)$ ... need $y'(4)$

\[
y'(4) = S(4) - R(4) = -1.909 \text{ yd}^3/\text{hr}
\]

This means Sandy Point beach is losing \approx 2 yd$^3$ of sand per hour at $t = 4$ hrs.

d. For $0 \leq t \leq 6$, at what time $t$ is the amount of sand on the beach a minimum? What is the minimum value? Justify your response.

\[
y(t) = 2500 + \int_0^t [S(x) - R(x)] \, dx
\]

$y'(t) = 0$ when $S(x) - R(x) = 0$

\[
y'(t) = 0 \quad \text{when} \quad S(x) - R(x) = 0
\]

\[
\begin{array}{c|c|c|c|c}
\hline
\cellcolor{white} & y(t) & t & y(t) \\
\hline
0 & 2500 & 0 & 2500 \hline
\text{A} & 2492.369 & 4 & \text{B} \hline
6 & 2493.277 & \hline
\end{array}
\]

The minimum amount of sand is 2492.369 yd$^3$ and the minimum occurs at $t = 5.118$ hrs.
Example 4

The rate at which wizards pass through the wall onto Platform $9 \frac{3}{4}$, measured in wizards per minute, is modeled by the function $F(t) = 24 + 2\sin\left(\frac{t}{2}\right)$ for times the platform is open $0 \leq t \leq 30$ in minutes. There are initially 50 wizards on the platform. The rate of wizards leaving the platform is represented by $G(t) = 3 + \frac{6}{t+4}$ also for $0 \leq t \leq 30$ in minutes.

a) To the nearest whole number, how many wizards enter onto Platform $9 \frac{3}{4}$ in the 30 minute period?

$\int_{0}^{30} F(t) \, dt = 727.039$ wizards

b) Is the amount of wizards on the platform increasing or decreasing at $t = 10$ minutes?

$Y(t) = 50 + \int_{0}^{t} [F(x) - G(x)] \, dx$

$Y'(10) = F(10) - G(10) = 18.654 > 0$ so increasing

c) How many total wizards are on the platform at $t = 30$ minutes?

$Y(30) = 50 + \int_{0}^{30} [F(t) - G(t)] \, dt = 1674.198$ wizards
8.1 AP Questions (Calculator Allowed on all)

2003 AB2 Form B
2002 AB2 Form B
2004 AB2 Form B
2010 AB1
2006 AB2
2005 AB2 Form B
2004 AB1
8.2 Areas in the Plane

Topics

- Areas between Curves
- Area Enclosed by Intersecting Curves
- Boundaries with changing functions

Warm Up!

Find the area between the x-axis and the graph of the given function over the given interval without a calculator.

a. \( y = \sin x \) over \([0, \pi]\)

\[
\int_0^\pi \sin x \, dx = \cos x \bigg|_0^\pi = \cos \pi - \cos 0 = -1 - 1 = -2 = \boxed{2}
\]

Find the x- and y-coordinates of all points where the graphs of the given functions intersect without a calculator.

b. \( y = \frac{2x}{x^2 + 1} \) and \( y = x^3 \)

\[
\frac{2x}{x^2 + 1} = \frac{x^3}{1}
\]

\[
2x = x^5 + x^3
\]

\[
0 = x^5 + x^3 - 2x
\]

\[
0 = x(x^4 + x^2 - 2)
\]

Recall Riemann Sums

The area under a curve can be approximated through the use of Riemann sums: We can break the area into rectangles. Consider the one rectangle drawn. It’s height is given by the function value of the curve at the right endpoint and the width is given as \( \Delta x \). The area under the curve then is approximately the sum of the areas of ALL the rectangles just like this one.
Recall Riemann Sums
As the number of rectangles, $n$, increases, the approximated area gets closer to the actual area, so we say

\[
\text{Area under the curve} = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \Delta x_k = \int_{a}^{b} f(x) \, dx
\]

We can apply this same concept to the area between curves. Consider the two functions $f$ and $g$ below.

Example 1
Draw a rectangular strip. What is the height and width of your rectangle? Would the height and width of the rectangle strip be different if you drew it in a different place?

The area between the curves is approximately the sum of all these rectangles. We can write this as

\[
\sum_{k=1}^{n} [f(x_k) - g(x_k)] \Delta x
\]

How can we get closer to the ACTUAL area between the curves?

make more rectangles!

If we let the number of rectangles approach infinity, then we have

\[
\lim_{n \to \infty} \sum_{k=1}^{n} [f(x_k) - g(x_k)] \Delta x = \int_{a}^{b} [f(x) - g(x)] \, dx
\]
Example 2: Area Between Two Curves

Find the area of the region between the two curves.

a. \( f(x) = 2 - x^2 \) and \( g(x) = x \)

\( \text{Points of intersection} \)

\[ 2 - x^2 = x \]
\[ 0 = x^2 + x - 2 \]
\[ 0 = (x + 2)(x - 1) \]
\[ x = -2, x = 1 \]
\[ (-2, -2), (1, 1) \]

\[ \int_{-2}^{1} (f(x) - g(x)) \, dx = \int_{-2}^{1} (2 - x^2 - x) \, dx \]

\[ = \left[ 2x - \frac{1}{3}x^3 - \frac{1}{2}x^2 \right]_{-2}^{1} \]

\[ = \left( 2 - \frac{1}{3} - \frac{1}{2} \right) - \left( -4 + \frac{8}{3} + 2 \right) \]

\[ = \frac{7}{6} + \frac{10}{3} = \frac{9}{2} \]

b. \( y = x^2 \) and \( y = e^x \) on the interval \([0, 3]\).
c. \( f(x) = 3x^3 - x^2 - 10x \) and \( g(x) = -x^2 + 2x \)

\[ 3x^3 - x^2 - 10x = -x^2 + 2x \]
\[ 3x^3 - 12x = 0 \]
\[ 3x(x^2 - 4) = 0 \]
\[ x = 0 \]
\[ x = \pm 2 \]

Pts. of intersection
\( (0, 0), (2, 0), (-2, -8) \)

\[ \text{top - btm} \]
\[ \int_{-2}^{2} (3x^3 - x^2 - 10x - (-x^2 + 2x)) \, dx \]
\[ + \int_{0}^{2} (x^2 + 2x - (3x^3 - x^2 - 10x)) \, dx \]
\[ 12 + 12 = 24 \]

d. In Quadrant 1: \( y = \sqrt{x} \), \( y = x - 2 \), and the x-axis.

\[ \int_{0}^{2} \sqrt{x} \, dx + \int_{2}^{4} [\sqrt{x} - (x - 2)] \, dx \]
\[ = \frac{10}{3} \]

or

\[ y = \sqrt{x} \]
\[ x = y^2 \]

\[ \int_{0}^{2} [(y + 2) - y^2] \, dy \]
\[ = \frac{10}{3} \]
e. \( y = x^2 + 2, y = -x, x = 0, \) and \( x = 1 \)

\[
\int_0^1 [(x^2 + 2) - (-x)] \, dx \\
\int_0^1 (x^2 + x + 2) \, dx \\
\frac{1}{3} x^3 + \frac{1}{2} x^2 + 2x \bigg|_0^1 \\
= \frac{1}{3} + \frac{1}{2} + 2 = \frac{17}{6}
\]

f. \( x = 3 - y^2 \) and \( x = y + 1 \)

\[
\int_{-2}^2 [(3-y^2) - (y+1)] \, dy \\
\int_{-2}^2 (2-y-y^2) \, dy = 2y - \frac{1}{2}y^2 - \frac{1}{3}y^3 \bigg|_{-2}^2 \\
\left[2(1) - \frac{1}{2}(1)^2 - \frac{1}{3}(1)^3\right] - \left[2(-2) - \frac{1}{2}(-2)^2 - \frac{1}{3}(-2)^3\right] \\
2 - \frac{1}{2} - \frac{1}{3} - (-4 - 2 + \frac{8}{3}) \\
= \frac{9}{2}
\]
8.3 Volumes

Topics

- Volume as an Integral
- Square Cross Sections
- Circular Cross Sections
- Cylindrical Shells
- Other Cross Sections

Warm Up!

Give a formula for the area of the plane region in terms of the single variable $x$.

a. A square with sides of length $x$

$$A = x^2$$

b. A square with diagonals of length $x$

$$A = \frac{x}{\sqrt{2}} \cdot \frac{x}{\sqrt{2}} = \frac{x^2}{2}$$

$$A = \frac{x^2}{2}$$

c. A semicircle of radius $x$

$$A = \frac{1}{2} \pi (x)^2 = \frac{\pi x^2}{2}$$

d. A semicircle of diameter $x$

$$A = \frac{1}{2} \pi \left(\frac{x}{2}\right)^2 = \frac{1}{8} \pi x^2$$

$$A = \frac{\pi x^2}{8}$$

e. An equilateral triangle with sides of length $x$

$$A = \frac{1}{2} (x) \left(\frac{x\sqrt{3}}{2}\right)$$

$$A = \frac{x^2\sqrt{3}}{4}$$

f. An isosceles right triangle with legs of length $x$

$$A = \frac{1}{2} x^2$$

Just like in the last section where we found the area of one arbitrary rectangular strip and used an integral to add up the areas of an infinite number of infinitely thin rectangles, we are going to apply the same concept to finding volume. The key... find the volume of ONE arbitrary slice and use an integral to add up the volumes of an infinite number of infinitely thin slices.
Day 1: Volumes of Solids with Known Cross Sections (aka \( \perp \))

First question… what is a cross section? Imagine a loaf of bread. Now imagine the shape of a slice through the loaf of bread. This shape would be a cross section. Technically a cross section of a three-dimensional figure is the intersection of a plane and that figure. It would be like cutting an object and looking at the face of where you just cut.

The cross sections we will be dealing with are almost entirely perpendicular to the x-axis.

Here’s the basic idea… you will be given a region defined by a number of functions. We will graph that region on an x and y-axis. Then we will lay the region flat and build upon that region which has the same cross section no matter where you slice it.

Second question… How do we find the volume of this slice that has been created to have a similarly shaped cross section, even though each cross section may have a different size? We get to use Calculus, of course! But first, we need to know how to find the volume of a prism. Even though every shape may be different, we can find the volume of a prism by finding the area of the base times the “height”. The “height” of our prisms will be the thickness of the slices. Once you know the volume of one slice, you just use an integral to add the volumes of all the slices to get the volume of the solid.

Example 1

a. Find the volume of the following square “slice”. Since most of the “slices” we will be dealing with will have a thickness of \( dx \), we will use that same thickness here.

\[
V = x \cdot x \cdot dx = x^2 dx
\]

Vol. of 1 slice

\[
V = \text{(Area of slice)} \times \text{(Thickness of slice)}
\]

b. Find the volume of the following semicircular “slice”

\[
A = \frac{1}{2} \pi \left( \frac{x}{2} \right)^2 dx = \frac{\pi x^2}{8} dx
\]

\[
A = \frac{\pi x^2}{8} dx
\]
Example 2
The base of a solid is the region in the first quadrant enclosed by the parabola \( y = 4x^2 \), the line \( x = 1 \), and the x-axis. Each plane section of the solid perpendicular to the x-axis is a square. The volume of the solid is

\[
V_{\text{slice}} = (\text{area}) \cdot \text{thickness} = (\text{side})^2 \, dx
\]

\[
V_{\text{slice}} = (4x^2)^2 \, dx = 16x^4 \, dx
\]

\[
V_{\text{total}} = \int_0^1 16x^4 \, dx = 16 \frac{x^5}{5} \bigg|_0^1 = \frac{16}{5} - \frac{16(0)}{5} = \frac{16}{5}
\]

\[\text{a.} \quad \frac{4\pi}{3} \quad \text{b.} \quad \frac{16\pi}{5} \quad \text{c.} \quad \frac{4}{3} \quad \text{d.} \quad \frac{16}{5} \quad \text{e.} \quad \frac{64}{5}\]

Example 3
The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis, and the line \( x + 2y = 8 \). If the cross sections of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid?

\[
V_{\text{slice}} = \text{area} \cdot \text{thickness}
\]

\[
V_{\text{slice}} = \frac{1}{2} \pi r^2 \cdot dx
\]

\[
V_{\text{slice}} = \frac{1}{2} \pi \left(\frac{1}{4} x + 4\right)^2 \cdot dx
\]

\[
V_{\text{total}} = \int_0^8 \frac{1}{2} \pi \left(\frac{1}{4} x + 4\right)^2 \, dx
\]

\[= 16.755\]

\[\text{a.} \quad 12.566 \quad \text{b.} \quad 14.661 \quad \text{c.} \quad 16.755 \quad \text{d.} \quad 67.021 \quad \text{e.} \quad 134.041\]
Example 4
The base of a solid is the region in the first quadrant enclosed by the graph of \( y = 2 - x^2 \) and the coordinate axes. If every cross section of the solid perpendicular to the \( y \)-axis is a square, the volume of the solid is given by

a. \( \pi \int_0^2 (2 - y)^2 \, dy \)

b. \( \int_0^2 (2 - y) \, dy \)

c. \( \pi \int_0^{\sqrt{2}} (2 - x^2)^2 \, dx \)

d. \( \int_0^{\sqrt{2}} (2 - x^2)^2 \, dx \)

e. \( \int_0^{\sqrt{2}} (2 - x^2) \, dx \)

Day 2: Volumes of Solids of Revolution: The Disc Method
In finding the area of a region we drew an arbitrary representative rectangle. Keep with the same idea, if we revolve a rectangle around a line, it forms a \( \text{cylinder} \), as shown below.

Keys to Disc Method
1. Always rotate around an \( \underline{\text{edge}} \) of the region; it must touch the region.
2. “Slice” or rectangle is \( \underline{\perp} \) to axis or revolution creating a cylinder.

Example 5
What is the volume of the cylinder shown if the height of the rectangle is considered \( R \) and the width of the rectangle is \( dx \)?

\[
V = \text{Area} \cdot \text{thickness} = \pi (R)^2 \, dx
\]
Just like we did in finding the area, as we increase the number of rectangles to infinity, the width of each rectangle becomes infinitely small and we denote this $dx$ (if it is a vertical strip) or $dy$ (if it is a horizontal strip). We then use an integral to sum the volume of every one of these infinitely thin cylinders.

**The Disc Method**

$V_{\text{slice}} = \pi r^2 \cdot \text{thickness}$

To find the volume of a solid of revolution with the disc method, use one of the following:

- **Horizontal Axis of Revolution**
  
  $V = \pi \int_a^b [R(x)]^2 \, dx$

- **Vertical Axis of Revolution**
  
  $V = \pi \int_a^b [R(y)]^2 \, dy$

Where $R(x)$ and $R(y)$ are the **heights** of your representative rectangular strips

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**Example 6**

Draw an appropriate rectangular strip and find the volume of the solid formed by revolving the region about the x-axis. (www.shodor.org/interactive/activities/functionrevolution/)

\[ V_{\text{slice}} = \pi R^2 \cdot dx \]

\[ = \pi (4-x^2)^2 \, dx \]

\[ = \pi \left(16 - 8x^2 + x^4\right) \, dx \]

\[ V_{\text{solid}} = \pi \int_0^2 \left(16 - 8x^2 + x^4\right) \, dx \]

\[ = \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{0}^{2} \]

\[ = \pi \left[16(2) - \frac{8(2)^3}{3} + \frac{(2)^5}{5}\right] - \pi \left[0 - 0 + 0\right] \]

\[ = \pi \left[32 - \frac{64}{3} + \frac{32}{5}\right] \]

\[ V_{\text{solid}} = \frac{256\pi}{15} \]
Example 7
Find the volume of the solid formed by revolving the region about the y-axis. (Draw a representative rectangular strip)

\[ V_{\text{slice}} = \pi R^2 \, dy \]
\[ = \pi (\sqrt{16-y^2})^2 \, dy \]
\[ = \pi (16-y^2) \, dy \]

\[ V_{\text{solid}} = \pi \int_0^4 (16-y^2) \, dy \]
\[ \pi \left[ 16y - \frac{1}{3} y^3 \right]_0^4 \]
\[ = \frac{128 \pi}{3} \]

Example 8
Find the volume of the solid generated by revolving the region bounded by the graphs of the equations \( xy = 6, y = 2, y = 6 \) and \( x = 6 \) about the indicated lines. Sketch the region formed, and draw a representative rectangular strip for each solid.

a. the line \( x = 6 \)

\[ R_{\text{right-left}} = \frac{6}{y} \]
\[ R_{\text{left-right}} = 6 - \frac{6}{y} \]

\[ V_{\text{slice}} = \pi R^2 \, dy \]
\[ = \pi \left( \frac{6}{y} \right)^2 \, dy \]
\[ V_{\text{solid}} = \pi \int_2^6 \left( \frac{6}{y} \right)^2 \, dy \]
\[ \approx 241.588 \]

b. the line \( y = 6 \)

\[ R_1 = 6 - \frac{6}{x} \]
\[ R_2 = 6 - y \]
\[ V_{\text{slice}_1} = \pi \left( \frac{6}{x} \right)^2 \, dx \]
\[ V_{\text{slice}_2} = \pi \left( \frac{6}{6} \right)^2 \, dx \]
\[ = 48 \pi \]

\[ V_{\text{total}} = 48 \pi + \pi \int_0^3 \left( \frac{6}{x} \right)^2 \, dx \]
\[ \approx 203.887 \]
Example 9

a. Sketch the figure formed by rotating the rectangle around the given line. Do you see why it’s called the washer method?

b. What is the volume of the figure formed above?

\[ V_{\text{outer cylinder}} - V_{\text{inner cylinder}} \]

\[ \pi (8^2 \cdot 2) - \pi (3^2 \cdot 2) \]

\[ \pi (8^2 - 3^2) \cdot 2 \text{ thickness} \]

\[ \pi (R^2 - r^2) \, dx \]
We will call the outer radius $R$, and we will call the inner radius $r$. The height of the cylinder formed is just the \text{width} of the strip. Just like before, if we have an infinitely thin strip, this distance will be denoted $dx$ (if it is a vertical strip) and $dy$ (if it is a horizontal strip).

The volume of the solid formed by revolving a region around the axis using the **Washer Method** is given by

$$V = \pi \int_a^b [R^2 - r^2] \, dx$$

**Example 10**

Set up an integral, but do not solve to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the indicated lines.

$y = 2x^2; \quad y = 0; \quad x = 2$

\[x = \pm \sqrt{\frac{y}{2}}\]

\[R = 2, \quad r = \text{dist from y-axis} + 0\]

\[V_{\text{slice}} = \pi \left[ 2^2 - \left(\frac{y}{2}\right)^2 \right] dy\]

\[V_{\text{solid}} = \pi \int_0^8 \left[ y - \frac{y}{2} \right] dy\]

\[V_{\text{slice}} = \pi \int_0^8 \left[ 8^2 - (8-2x^2)^2 \right] dx\]

\[V_{\text{solid}} = \pi \int_0^8 \left[ 8^2 - (8-2\sqrt{\frac{y}{2}})^2 \right] dx\]
8.3 AP Questions
2001 AB1 (Calculator Allowed)
2002 AB1 (Calculator Allowed)
2007 AB1 (Calculator Allowed)
2012 AB2 (Calculator Allowed)
2013 AB5 (No Calculator)