Chapter 6: The Definite Integral

Sections:
- 6.1 Estimating with Finite Sums
- 6.2 Definite Integrals
- 6.3 Definite Integrals and Antiderivatives
- 6.4 Fundamental Theorem of Calculus

HW Sets
Set A (Section 6.1) Page 274-275, #’s 4, 5, 6, 16, 17.
Set B (Section 6.5) Page 316, #’s 3, 5, 7, 11.
Set C (Section 6.2) Pages 286 & 287 #’s 1-27 odd.
Set D (Section 6.3) Pages 294 & 295, #’s 1-5 odd, 15-25 odd.
Set E (Section 6.4) Page 306, #’s 1-23 odd.
Set F (Section 6.4) Page 307, #’s 27-47 odd.

LOOK AT ALL

THESE INTEGRALS I CAN’T SOLVE
Warm Up!
Suppose from the 2\textsuperscript{nd} to 4\textsuperscript{th} hour of your road trip, you travel with the cruise control set to exactly 70 miles per hour for that two-hour stretch.

a. How far have you traveled during this time?

b. Sketch a graph modeling $v(t) = 70$ mph. Is the total distance traveled represented on the graph? Explain?

Example 1: A Non-Constant Velocity
For each given velocity function, determine how far you have traveled from 0\textsuperscript{th} hour to the 4\textsuperscript{th} hour.

a. $v(t) = \frac{1}{2} t$

b. $v(t) = -|t - 2| + 2$

c. $v(t) = t^2$
The Area Problem and the Rectangular Approximation (RAM, a.k.a. Riemann Sums)

From example 1c, we need more techniques to approximate the area under the curve...

Suppose we wanted to know the area of the region bounded by a curve, the x-axis, and the lines $x = a$ and $x = b$, as shown to the right.

Step 1:

Step 2:

Step 3: Find height... three methods
  - Use _______ endpoint of each interval
  - Use _______ endpoint of each interval
  - Use the ______________ endpoint of each interval

Step 4:

Example 2: Looking at Graphs
Which method is shown in the two graphs below?

a.

Example 3
The total area under the curve then is approximately equal to the total area of all the rectangles. Which of the graphs above gives a better approximation of the area under the curve? Why? How could it be further improved?
Example 4
Illustrate the use of RRAM and MRAM on the graphs below (use 4 rectangles)

a. 

b. 

Example 5: Using Riemann Sums
Use the given method to approximate the area under the graph of \( y = x^2 \) from \( x = 1 \) to \( x = 3 \) using 4 subintervals. Identify if this is an over- or under-approximation.

a. \( \text{LRAM}_4 \)

b. \( \text{RRAM}_4 \)

c. \( \text{MRAM}_4 \)
Example 6: Using a Table of Velocity
It is not necessary to have a graph to estimate the area. Suppose the table below shows the velocity of a model train engine moving along a track for 10 seconds.

a. Using a left Riemann Sum with 10 subintervals, estimate the distance traveled by the engine in the first 10 seconds.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Velocity (in./sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

b. Using a Midpoint Riemann Sum with 5 subintervals, estimate the distance traveled by the engine in the first 10 seconds.
The Trapezoidal Rule (Section 6.5)

While rectangles make a fairly good approximation, it’s easy to see that we’re going to need a lot of them to provide a good estimate. We can find a better estimate in less time if we use _______________. If we were to _______________ the interval into subintervals like we did before, we can use each subinterval to create a trapezoid if we just connect the function values of the left and right endpoints. Before we begin, let’s make sure you remember the area formula for a trapezoid.

Area of a Trapezoid:

For the Trapezoid Rule, our trapezoid will look like:

Example 7: The Trapezoid Rule

Use 4 Trapezoids to approximate the area under the curve $y = x^2 - 2x + 2$ from $x = 1$ to $x = 3$. Sketch the trapezoids. Is this an over- or under-approximation?

The key to finding the total distance traveled in the last example in a method similar to the first example is to break the time intervals into such short segments, that the velocity over those time segments is almost constant (this will require LOTS of intervals). If the velocity is almost constant for each time interval, then we can find the distance traveled for each time interval (which is just the area of an extremely thin rectangle) and add all the areas of all the rectangles together. Sounds simple enough right? Can you guess what extremely important calculus concept is involved?
6.2 Definite Integrals

Topics

- Riemann Sums
- Terminology and Notation of Integration
- Definite Integral and Area
- Constant Functions
- Integrals on a Calculator
- Discontinuous Integrable Functions

Warm Up!
Could you determine the area under the curve from $x = a$ to $x = b$ for the following graphs? Explain?

a.

We need a generic process for finding the sums that allow us to deal with negative values.

### Riemann Sum

**Step 1:** Start with a ______________ function on a _________ interval.

**Step 2:** ___________ the interval into n subintervals. What is the width?

**Step 3:** In each subinterval, pick ____ number and call the number picked from the $k^{th}$ subinterval $c_k$.

**Step 4:** For each interval, create a rectangle that extends from the x-axis to the function value $f(c_k)$, of the number you picked in each interval. (Some of these rectangles may lie below the x-axis)

**Step 5:** Find the _____ of each product consisting of the function value at $c_k$ multiplied by the width of the interval.
**Riemann Sum for f on the interval [a, b]**

Any sum of the form

\[ \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \]

where \( f \) is a continuous function on a closed interval \([a, b]\), partitioned into \( n \) subintervals and where the \( k^{th} \) subinterval contains some point, \( c_k \), and had length \( \Delta x_k \).

Every Riemann sum depends on the partition you choose (i.e. the number of subintervals) and your choice of the number within each interval, \( c_k \).

**Example 1: Riemann Sum on a Closed Interval**

Discuss with your group...

a. Do the partitions have to be equal widths?

b. Does \( c_k \) have to be any particular value in the sub-interval?

c. When have we done this process (add up the product of lengths and widths) before?

d. Is our answer an exact answer or an approximation? If it is approximate, how do we make it “better”?

e. Could our answers for this sum be negative?

**Definition of a Definite Integral (or Riemann Integral)**

A DEFINITE INTEGRAL is simply the __________ of a Riemann Sum.

**Option #1:** From before, we didn’t care if our subintervals were the same width. If we use the notation \( ||P|| \) to denote the longest subinterval length, we can force the longest subinterval length to ___ using a limit of the Riemann Sum as follows:

\[ \lim_{||P|| \to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \]

**Option #2:** If we make sure the subintervals are all the same width, we can increase the number of rectangles to infinity using a limit of the Riemann Sum as follows:

\[ \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \]
Notation for Definite Integrals

We read the notation as “The Integral of f of x from a to b”

**If the function is ________________, THEN the Definite Integral will exist. However, the converse, while true some of the time is NOT ALWAYS TRUE**

Example 2: Limit Definition of the Riemann Sum.
Match the integral expression in the left column with the appropriate limit of a Riemann sum in the right column.

1. \( \int_1^3 (4x^2 + 2) \, dx \) 
   a. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ 4 \left( 5 - \frac{3j}{n} \right) - 2 \right] \left( \frac{3}{n} \right) \)

2. \( \int_2^5 (x^3 + 1) \, dx \) 
   b. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ 4 \left( 2 + \frac{3j}{n} \right)^2 + 2 \right] \left( \frac{3}{n} \right) \)

3. \( \int_7^5 (3x + 1) \, dx \) 
   c. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left( 2 + \frac{3j}{n} \right)^3 + 1 \) \left( \frac{3}{n} \right) \)

4. \( \int_2^4 (4x^2 + 2) \, dx \) 
   d. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ 4 \left( 1 + \frac{2j}{n} \right)^2 + 2 \right] \left( \frac{2}{n} \right) \)

5. \( \int_5^2 (4x - 2) \, dx \) 
   e. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ 3 \left( 7 - \frac{2j}{n} \right) + 1 \right] \left( \frac{-2}{n} \right) \)

6. \( \int_2^5 (4x^2 + 2) \, dx \) 
   f. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ 4 \left( 3 - \frac{2j}{n} \right)^2 + 2 \right] \left( \frac{-2}{n} \right) \)

7. \( \int_5^7 (4x - 2) \, dx \) 
   g. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ 3 \left( 2 + \frac{3j}{n} \right) + 1 \right] \left( \frac{3}{n} \right) \)

8. \( \int_3^1 (4x^2 + 2) \, dx \) 
   h. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ 4 \left( 5 + \frac{2j}{n} \right) - 2 \right] \left( \frac{2}{n} \right) \)

9. \( \int_5^7 (x^3 + 1) \, dx \) 
   i. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ \left( 5 + \frac{2j}{n} \right)^3 + 1 \right] \left( \frac{2}{n} \right) \)

10. \( \int_2^5 (3x + 1) \, dx \) 
    j. \( \lim_{n \to \infty} \sum_{j=1}^{n} \left[ 4 \left( 2 + \frac{2j}{n} \right)^2 + 2 \right] \left( \frac{2}{n} \right) \)
Using Definite Integrals as Area

We can define the area under the curve \( y = f(x) \) from \( a \) to \( b \) as an ___________ from \( a \) to \( b \)... as long as the curve is __________________ and _______________ on the closed interval \([a, b]\).

Drawing a picture and using geometry is still a valid method of finding areas in this class!

Example 3: Definite Integrals as Areas

For each of the following examples, sketch a graph of the function, shade the area you are trying to find, then use geometric formulas to evaluate each integral.

a. \( \int_{2}^{9} 3\,dx \)

b. \( \int_{-2}^{1} |x|\,dx \)

c. \( \int_{3}^{10} \sqrt{9 - x^2}\,dx \)

Example 4: A Definite Integral with Negative Values

Consider the function \( f(x) = 3 - x \).

a. Sketch a graph of this function.

b. What is the “Area” between the curve and the x-axis between \( x = 4 \) and \( x = 8 \)?

c. Evaluate \( \int_{4}^{8} (3 - x)\,dx \)
Example 5: Definite Integral Practice
Given $\int_0^{\pi} \sin x \, dx = 2$, use what you know about a sine function to evaluate the following integrals.

a. $\int_{\pi}^{2\pi} \sin x \, dx$  
b. $\int_0^{2\pi} \sin x \, dx$  
c. $\int_0^{\pi} \sin x \, dx$

d. $\int_{-\pi}^{\pi} \sin x \, dx$  
e. $\int_{\pi}^{\pi} (2 + \sin x) \, dx$  
f. $\int_0^{\pi/2} \sin x \, dx$

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Example 6: Using the Calculator to Evaluate Definite Integrals
Use your calculator to evaluate.

a. $\int_1^3 (x^2 - 2x + 2) \, dx$  
b. $3 + 2 \int_0^{\pi/3} \tan x \, dx$
Using Graphing to Calculate Definite Integrals

1. Graph the function
2. 2nd Trace
3. \( \int f(x) \, dx \)
4. Enter lower bound
5. Enter upper bound

The only down side to using this method is that you MUST be able to set your window to SEE everything.

Example 7: Using the Calculator to Evaluate Definite Integrals
Use your calculator to graph and evaluate.

a. \( \int_{1}^{8} \sqrt{x} \, dx \)

b. \( \int_{5}^{12} \sqrt{x - 5} \, dx \)

Example 8: PVA
The rate at which water flows into a tank, in gallons per hour, is given by a positive continuous function \( R \) of time \( t \). The table below shows the rate at selected values of \( t \) for a 12-hour period.

<table>
<thead>
<tr>
<th>( t ) (hrs)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(t) ) (gal/hr)</td>
<td>12.5</td>
<td>13.4</td>
<td>13.9</td>
<td>14.3</td>
<td>14.6</td>
<td>14.8</td>
<td>14.7</td>
</tr>
</tbody>
</table>

a. Use a midpoint Riemann Sum with three subintervals to approximate: \( \int_{0}^{12} R(t) \, dt \)
Example 9: PVA
Particle A moves along a horizontal line with a velocity $V_A(t)$, where $V_A(t)$ is a positive continuous function of $t$. The time $t$ is measured in seconds, and the velocity is measured in cm/sec. The velocity $V_A(t)$ of the particle at selected times is given in the table below.

<table>
<thead>
<tr>
<th>$t$ (sec)</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_A(t)$ (cm/sec)</td>
<td>1.7</td>
<td>6.8</td>
<td>7.4</td>
<td>15.6</td>
<td>24.9</td>
</tr>
</tbody>
</table>

a. Use data from the table to approximate the distance traveled by particle A over the interval $0 \leq t \leq 10$ seconds by using a right Riemann sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.
6.3 Definite Integrals and Antiderivatives

Topics

- Properties of Definite Integrals
- Average Value of a Function
- Mean Value Theorem for Definite Integrals
- Connecting Differential and Integral Calculus

Warm Up!

Find \( \frac{dy}{dx} \) without a calculator.

a. \( y = -\cos x \sin x \)

b. \( y = \ln(\sec x) \)

c. \( y = xe^x \)

d. \( y = \frac{1}{2^x+1} \)

Properties (Rules) for Definite Integrals

1. Order of Integration: \( \int_a^b f(x)dx = \)
   - If you reverse the order of integration, you get the opposite answer.

2. Zero: \( \int_a^a f(x) dx = \)
   - This should make sense if you think about the “area” of a rectangle with no width.

3. Constant Multiple: If \( k \) is any constant, then \( \int_a^b k \cdot f(x) dx = \)
   - Taking the constant out of the integral many times makes it simpler to integrate.

4. Sum and Difference: \( \int_a^b [f(x) \pm g(x)]dx = \)
   - Allows you to integrate functions that are added or subtracted separately. Notice, there are NO rules here for two functions that are multiplied or divided... that comes later!
Example 1: Using the Rules for Definite Integrals
Suppose $\int_2^6 f(x)dx = 10$ and $\int_2^6 g(x)dx = -2$, find the following.

a. $\int_2^6 [f(x) + g(x)]dx$

b. $\int_2^6 [g(x) - f(x)]dx$

c. $\int_2^6 3f(x)dx$

d. $\int_2^6 (f(x) + 2)dx$

Example 2: Using the Rules for Definite Integrals
Suppose $\int_{-1}^1 f(x)dx = 5$, $\int_{-1}^4 f(x)dx = -2$, and $\int_{-1}^1 h(x)dx = 7$, find the following.

a. $\int_4^1 f(x)dx$

b. $\int_0^1 f(x)dx$

c. $\int_{-2}^2 h(x)dx$

d. $\int_{-1}^4 f(x)dx$

e. $\int_{-1}^1 [2f(x) + 3h(x)]dx$

f. $\int_{-1}^1 [f(x) + h(x)]dx$

Properties for Definite Integrals cont…

5. Additivity: $\int_a^b f(x) dx + \int_b^c f(x)dx =$
   - Piece + Piece = Whole... Look at the lower and upper bounds of integration. This comes in handy when dealing with total area or other functions where we need to break them into smaller parts.
Example 3: Indefinite Integrals
Integrate

a. \( \int 3x^2 + 3 \)  
b. \( \int 3x^2 + 7 \)  
c. \( \int 2x^4 \)

Power Formulas for Indefinite Integrals
The family of all antiderivatives of a function \( f(x) \) is the indefinite integral of \( f \) with respect to \( x \) and is denoted \( \int f(x) \, dx \). If \( F \) is any function such that \( F'(x) = f(x) \), then \( \int f(x) \, dx = F(x) + C \), where \( C \) is called the ___________________________.

\[
\int u^n \, du =
\]

Example 4: Indefinite Integrals
Integrate

a. \( \int \frac{1}{x^3} \)  
b. \( \int \sqrt{x} \)  
c. \( \int 2 \sin x \, dx \)

d. \( \int dx \)  
e. \( \int (x + 2) \, dx \)  
f. \( \int (3x^4 - 5x^2 + x) \, dx \)

g. \( \int \frac{x+1}{\sqrt{x}} \, dx \)
Example 5: Using Differentiation to Check Antiderivative

Find the general solution of \( F'(x) \). Then find the particular solution that satisfies the initial condition provided.

a. \( F'(x) = \frac{1}{x^2}, x > 0; F(1) = 0 \)

b. \( \frac{dy}{dx} = 4x \) where my initial condition is that \( f(0) = 6 \)

c. \( \frac{dy}{dx} = 6x^2 \) where my initial condition is that \( f(0) = -1 \)
**6.4 Fundamental Theorem of Calculus**

**Topics**
- Fundamental Theorem
- Area Connection
- Analyzing Antiderivates Graphically

**Warm Up!**

Find $\frac{dy}{dx}$ without a calculator.

a. $y = \sin(x^2)$  
b. $y = 2^x$

c. $y = \ln(3x) - \ln(7x)$  
d. $y = \sqrt{x}$

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**The Fundamental Theorem of Calculus**

The FTOC has two parts. These two parts tie together the concept of integration and differentiation and is regarded by some to be the most important computational discovery in the history of mathematics!

**Example 1: The FTOC**

In section 6.2, we found an estimate for the distance traveled by finding the area between a velocity function and the x-axis. If $v(t)$ is that velocity function (that is above the x-axis) and the time is from 0 to 10 seconds...

a. How might we use calculus to define the distance traveled?

b. When $v(t) = 3t + 30$, evaluate your answer from part (a) using a graphing calculator.
Example 2: The FTOC
Suppose a car’s position is given by \( s(t) = \frac{3}{2} t^2 + 30t + 25 \) where \( t \) is time in seconds, and \( 0 \leq t \leq 10 \).

a. What is the position of the car at \( t = 0 \) seconds? What is the position of the car at \( t = 10 \) seconds?

b. What is the change in position of the car from time \( t = 0 \) to time \( t = 10 \) seconds?

c. How does this question relate to the previous example?

d. Just to double check... find \( s'(t) \)... notice anything???

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The Fundamental Theorem of Calculus (The Evaluation Part)
We no longer need to evaluate integrals graphically!

If \( f \) is continuous at every point of \([a, b]\),

\[
\int_a^b f(x) \, dx =
\]

Where \( F(x) \) is an \( \underline{\text{___________}} \) of \( f(x) \).

Now an integral is the difference of antiderivative of \( f(x) \) evaluated at upper bound and then lower bound...
Example 3: The FTOC

Use the evaluation part of the FTOC to evaluate each expression:

a. \[ \int_{0}^{3} x^2 \, dx \]  
   b. \[ \int_{\pi/2}^{\pi} (1 + \cos x) \, dx \]

   c. \[ \int_{-1}^{3} (x^3 + 1) \, dx \]  
   d. \[ \int_{-1}^{2} 3x \, dx \]

   e. \[ \int_{-1}^{3} x^3 \, dx \]  
   f. \[ \int_{1}^{32} x^{-6/5} \, dx \]
This last example will be an extremely important concept as we go through the rest of the year.

Using the evaluation part, we are going to develop the concept of the other part of the FTOC. Your book calls this Part 1, because it proves them in the opposite order. Our goal here isn’t really to prove the FTOC, Part 1, but to understand how it works.

We are going to determine how to take a derivative of a function that is defined as an integral and discuss what it means to define a function as an integral. Once we can do both of these things, we can answer all the same types of questions about increasing, decreasing, concave up, concave down, and inflection points that we did earlier in the year.
The Second Fundamental Theorem of Calculus

If \( f \) is continuous on \([a, b]\), then the function

\[
F(x) = \int_a^x f(t) \, dt
\]

has a derivative at every point \( x \) in \([a, b]\), and

\[
\frac{dF}{dx} = \frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

1. Derivative of an Integral.
2. Derivative matches upper limit of integration.
3. Lower limit of integration is a constant.

Example 4: Using the 2nd FTOC

Find the following by using the Fundamental Theorem.

a. \( \int_{-\pi}^x \cos t \, dt \)

b. \( \int_0^x (2t - 5) \, dt \)

c. \( \int_3^x t^5 \, dt \)

d. \( \int_3^{x^2} t^5 \, dt \)

e. \( \frac{d}{dx} \int_{-\pi}^x \cos t \, dt \)

f. \( \frac{d}{dx} \int_0^x (2t - 5) \, dt \)

g. \( \frac{d}{dx} \int_3^x t^5 \, dt \)

h. \( \frac{d}{dx} \int_3^{x^2} t^5 \, dt \)

i. \( \frac{d}{dx} \int_{x^3} t^5 \, dt \)

j. \( \frac{d}{dx} \int_{x^2}^{3} t^5 \, dt \)
Example 5: Using the 2nd FTOC with Shortcut

Simplify

a. \( \frac{d}{dx} \int_{1}^{x} \sqrt[3]{t} \, dt \)

b. \( \frac{d}{dx} \int_{x}^{4} 2t \, dt \)

c. \( \frac{d}{dx} \int_{3}^{x} \cos t \, dt \)

d. \( \frac{d}{dx} \int_{x}^{x^2} t^5 \, dt \)

e. \( \frac{d}{dx} \int_{1}^{\cos x} 2t \, dt \)

f. \( \frac{d}{dx} \int_{x}^{x^2} \frac{t^3}{t^4+1} \, dt \)

Extra Practice: Pages 320 & 321, #’s 9, 13, 14, 15-23 odd, 26, 27, 31, 39-42, 45 w/ calc.
**Warm Up!**

Suppose you wanted to find the average temperature during a 24-hour period.

a. How could you do it?

b. Suppose \( f(t) \) represents the temperature at time \( t \), measure in hours since midnight. One way to start is to measure the temperature at \( n \) equally spaced times \( t_1, t_2, t_3, \ldots, t_n \) and average those temperatures. Using this method, write an expression for the average temperature.

c. The larger number of measurements, the more accurately this will reflect the average temperature. Notice we can write this expression as a Riemann sum by first noting that the interval between measurements will be \( \Delta t = \frac{24}{n} \), so \( n = \)

d. Substitute this value for \( n \) into your expression above and simplify.

e. The last expression gives us an approximate Average Temperature. As \( n \to \infty \) (meaning we are taking a lot of temperature readings) this Riemann Sum becomes a __________ ____________. Write the __________ ____________ that gives us the Average Temperature since midnight.

d. Do you think that there is any point during the day that the temperature reading on the thermometer is the exact value of the average temperature?
**Average Value of a Function**

The process that we just used to find the average temperature is used to find the Average Value of any function.

If \( f \) is integrable on \([a, b]\), its average value (___ value) on \([a, b]\) is given by

\[
\text{AVERAGE VALUE} = \quad \ldots \text{ or } \ldots =
\]

The average value of a function is just... “The ______ over the ______.”

\[
\frac{\text{Sum}}{\text{length of interval}}
\]

---

**Example 1: Geometric Representation of Average Value of a Function**

Use the function \( y = x^2 \) on \([0, 3]\).

a. Graph the function on the grid provided.

b. Set up a definite integral to find the average value of \( y \) on \([0, 3]\), then use your calculator to evaluate the definite integral.

c. Graph this value as a function on the grid to the right. Does the function ever actually equal this value? If so, at what point(s) in the interval does the function assume its average value?

d. What do you suppose is the relationship between the area between the x-axis and the curve \( y = x^2 \) on \([0, 3]\) and the area of the rectangle formed using the average value as the height and the interval \([0, 3]\) as the width?
**Mean Value Theorem for Definite Integrals**

The Mean Value Theorem for Integrals basically says that if you are finding the area under a curve between \( x = a \) and \( x = b \), then there is some number \( c \) between \( a \) and \( b \) whose function value you can use to form a rectangle that has an area _________ to the area under the curve.

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**Example 2: Mean Value Theorem for Definite Integrals**

Use the diagram provided to answer.

a. What is an expression that could be used to determine the area under the curve from \( a \) to \( b \)?

b. What is the area of the shaded rectangle?

c. This value of \( f(c) \) is just the _________ _________ of \( f \) on the interval \([a, b]\).

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**Mean Value Theorem for Integrals**

So another way to look at this is the MVT for integrals just says that at some point within the interval, the function MUST equal its average value.

If \( f \) is continuous on \([a, b]\), then at some point \( c \) in \([a, b]\),

\[
f(c) =
\]

... Once again, we have a theorem that tells us a value of \( c \) exists, but the theorem doesn’t actually find it for us!

It is greatly important that you understand the difference between average rate of change and average value.

More on this after we finish 6.4. For now, understand that the average rate of change is simply the “slope between two points” on a given function and the average value of the function is the “integral divided by the interval”