All work must be shown in this course for full credit. Unsupported answers may receive NO credit.

1. The only way to guarantee the existence of a limit is to algebraically prove it. Describe the different ways you can investigate the existence of a limit.

   Limits can be investigated graphically by seeing where the function is going or numerically using a table of values.

2. Using words, explain what is meant by the expression \( \lim_{q \to c} f(q) = L \).

   \( \lim_{q \to c} f(q) = L \) means ... as \( q \) gets very close to \( c \), the function value at \( q \) gets very close to \( L \).

3. How do you find the average speed of an object?

   \[
   \text{avg. speed} = \frac{\text{total distance covered}}{\text{elapsed time}} = \frac{\Delta \text{ in dist.}}{\Delta \text{ in time}}
   \]

   avg. speed is always positive

4. Suppose an object moves along the x-axis with its position function given by \( x(t) = 5t^2 + 7t \), where \( t \) is measured in seconds.

   a) What is the average speed from \( t = 2 \) to \( t = 4 \) seconds?

   \[
   \text{avg. speed} = \frac{x(4) - x(2)}{4 - 2} = \frac{108 - 34}{4 - 2} = 37 \text{ units/sec}
   \]

   b) How fast is the object moving at exactly \( t = 4 \) seconds? Let's investigate the speed as we get close to 4 seconds...

   avg. speed between \( t = 3.5 \) and \( t = 4 \) sec \( \Rightarrow \frac{x(4) - x(3.5)}{4 - 3.5} = 44.5 \)

   avg. speed between \( t = 3.9 \) and \( t = 4 \) sec \( \Rightarrow \frac{x(4) - x(3.9)}{4 - 3.9} = 46.5 \)

   avg. speed between \( t = 3.999 \) and \( t = 4 \) sec \( \Rightarrow \frac{x(4) - x(3.999)}{4 - 3.999} = 46.995 \)

   avg. speed appears to approach 47 units/sec.

5. An rover on another planet drops an object off a cliff. The object falls \( y = gt^2 \) m in \( t \) sec, where \( g \) is a constant. Five seconds after the object was dropped it lands 30 m below.

   a) Find the value of \( g \).

   \[
   \frac{30}{2\cdot5} = 9
   \]

   \[ y = gt^2 \quad y = 9 \]

   b) Find the average speed for the fall.

   \[
   \text{avg speed} = \frac{\Delta \text{dist}}{\Delta \text{time}} = \frac{y(5) - y(0)}{5 - 0} = \frac{30 - 0}{5} = 6 \text{ m/sec}
   \]

   c) With what speed did the rock hit the bottom? Investigate as rock gets very close...

   \[
   \approx \frac{y(5) - y(4.9999)}{5 - 4.9999} \approx 11.9998 \text{ m/sec about 12 m/sec}
   \]

6. Assume \( \lim_{x \to b} f(x) = 7 \) and \( \lim_{x \to b} g(x) = -3 \).

   a) \( \lim_{x \to b} (f(x) + g(x)) = \lim_{x \to b} f(x) + \lim_{x \to b} g(x) \)

   \[ = 7 + (-3) = 4 \]

   b) \( \lim_{x \to b} (f(x) \cdot g(x)) = \lim_{x \to b} f(x) \cdot \lim_{x \to b} g(x) \)

   \[ = 7 \cdot -3 = -21 \]

   c) \( \lim_{x \to b} 4g(x) = 4 \cdot \lim_{x \to b} g(x) \)

   \[ = 4 \cdot (-3) = -12 \]

   d) \( \lim_{x \to b} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to b} f(x)}{\lim_{x \to b} g(x)} \)

   \[ = \frac{7}{-3} = -\frac{7}{3} \]
7. When asked to evaluate the limit of a function, what should be done first?

To evaluate a limit, first try direct substitution.

8. Evaluate the following limits by using direct substitution.

   a) \( \lim_{x \to \pi} \sec \left( \frac{\pi x}{6} \right) = \frac{1}{\cos \left( \frac{\pi}{6} \right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \)

   b) \( \lim_{x \to -4} \frac{\sqrt{x} + 4}{x + 4} = \frac{2}{\sqrt{8}} = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2} \)

   c) \( \lim_{x \to 2} 3x^2(2x - 1) = 3 \left( \frac{1}{2} \right)^2 \left( 2 \cdot \frac{1}{2} - 1 \right) = 0 \)

   d) \( \lim_{y \to 2} \frac{y^2 + 5y + 6}{y + 2} = \frac{2^2 + 5(2) + 6}{2 + 2} = \frac{10}{4} = \frac{5}{2} \)

   e) \( \lim_{x \to -6} \sqrt{x} = \sqrt{-6} = \sqrt{-6} \)

   f) \( \lim_{x \to 2} \sqrt{x + 3} = \sqrt{2 + 3} = \sqrt{5} \)

9. Explain why you cannot use direct substitution to determine each of the following limits.

   a) \( \lim_{x \to -2} \sqrt{x - 2} = \sqrt{-2 - 2} = \sqrt{-4} \) not defined

   We cannot substitute \( x = -2 \) into \( \sqrt{x - 2} \) because \( -2 \) is not in the domain.

   b) \( \lim_{x \to 0} \frac{1}{x^2} = \frac{1}{0^2} = \frac{1}{0} \) not defined

   c) \( \lim_{x \to 0} \frac{4 + x^2 - 16}{x} = \lim_{x \to 0} \frac{(4 + x^2) - 16}{x} = \frac{0}{0} \) indeterminate

   y = \frac{1}{x} has VA at \( x = 0 \).

   As \( x \to 0 \), \( y \)-values grow without bound.

   'As we will learn other methods to determine this limit, we cannot substitute \( x = -2 \) into \( \sqrt{x - 2} \).

10. If a limit does not exist, there are 3 possible reasons why. List all three possible reasons why a limit may not exist.

   1. \( \lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x) \)

   2. \( f(x) \) increases or decreases without bound as \( x \) approaches \( c \).

   3. \( f(x) \) oscillates between two fixed values as \( x \) approaches \( c \).

11. Find each limit, or explain why the limit does not exist.

   a) \( \lim_{x \to 2} f(x) \), if \( f(x) = \begin{cases} \ln x & \text{for } 0 < x \leq 2 \\ \frac{x^2}{\ln x} & \text{for } 2 < x \leq 4 \end{cases} \)

   \( \lim_{x \to 2^-} f(x) = \ln 2 \)

   \( \lim_{x \to 2^+} f(x) = 2^\ln 2 \)

   Limit DNE because \( \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \)

   At \( x = 1 \), \( y = \frac{x^2 - 4}{x - 1} \) has VA.

   'As the limit DNE because the function grows without bound.

   b) \( \lim_{x \to \infty} f(x) \), if \( f(x) = 3x + 1 \), \( x < 2 \)

   \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x + 1}{x + 1} = \lim_{x \to \infty} \frac{3x}{x} + \frac{1}{x} = \lim_{x \to \infty} \frac{3}{1} = 3 \)

   At \( x = 2 \), \( y = \frac{x + 1}{x^2 - 4} \) has a vertical asymptote.

   'As the limit DNE because the function grows without bound.
12. Determine whether each statement about the graph below is True or False.

a) \( \lim_{{x \to 1^{-}}} f(x) = 1 \) \hspace{1cm} True  

b) \( \lim_{{x \to 2}} f(x) \) does not exist  

\( \lim_{{x \to 2}} f(x) = 1 \) \hspace{1cm} False  

d) \( \lim_{{x \to 1}} f(x) = 2 \) \hspace{1cm} True  

e) \( \lim_{{x \to 1^+}} f(x) = 2 \) \hspace{1cm} False  

f) \( \lim_{{x \to 1^-}} f(x) \) does not exist  

\( \lim_{{x \to 1^-}} f(x) \neq \lim_{{x \to 1^+}} f(x) \) \hspace{1cm} True  

g) \( \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} f(x) \) \hspace{1cm} True  

h) \( \lim_{{x \to c}} f(x) \) exists at every \( c \) in \( (-1, 1) \) \hspace{1cm} True  

i) \( \lim_{{x \to 3}} f(x) \) exists at every \( c \) in \( (1, 3) \) \hspace{1cm} True  

13. Use the graph of \( f(x) \) to estimate the limits and value of the function, or explain why the limit does not exist.

a) \( \lim_{{x \to 1^-}} f(x) = 2 \) \hspace{1cm} e) \( \lim_{{x \to 2}} f(x) = 3 \)  

b) \( \lim_{{x \to -1}} f(x) = -1 \) \hspace{1cm} f) \( \lim_{{x \to 2}} f(x) = 3 \)  

c) \( \lim_{{x \to 1}} f(x) \) DNE because \( \lim_{{x \to 1^-}} f(x) \neq \lim_{{x \to 1^+}} f(x) \)  

g) \( \lim_{{x \to 3}} f(x) = 3 \) \hspace{1cm} h) \( f(2) = 3 \)  

14. For each of the following functions, (i) draw the graph, (ii) determine \( \lim_{{x \to c^-}} f(x) \) and \( \lim_{{x \to c^+}} f(x) \), and (iii) explain what the value of \( \lim_{{x \to c^-}} f(x) \) is or explain why it doesn’t exist.

a) \( c = 2, \ f(x) = \begin{cases} 6 - x, & \text{if } x < 2 \\ 4, & \text{if } x = 2 \\ \frac{x}{2} + 3, & \text{if } x > 2 \end{cases} \) \hspace{1cm} b) \( c = -1, \ f(x) = \begin{cases} \frac{1-x^2}{x}, & \text{if } x \neq -1 \\ 3, & \text{if } x = -1 \end{cases} \)  

\( i \) see attached graph  

\( ii \) \( \lim_{{x \to 2^+}} f(x) = 4 \) \hspace{1cm} \( \lim_{{x \to 2^-}} f(x) = \frac{x}{2} + 3 = 4 \) \hspace{1cm} \( i \) see attached graph  

\( iii \) \( \lim_{{x \to 2}} f(x) = 4 \)  

\( ii \) \( \lim_{{x \to -1^+}} f(x) = 0 \) \hspace{1cm} \( \lim_{{x \to -1^-}} f(x) = 1 - (-1)^2 = 0 \) \hspace{1cm} \( iii \) \( \lim_{{x \to -1}} f(x) = 0 \)
15. Suppose \( f(x) =\begin{cases} \sqrt{1-x^2}, & \text{if } 0 \leq x < 1 \\ 3, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } x = 2 \end{cases}\). Draw a graph of \( f(x) \), then answer the following questions.

See graph on attached pg.

a) At what points \( c \) in the domain of \( f \) does \( \lim_{x \to c} f(x) \) exist?

\[ \lim_{x \to c} f(x) \text{ exists when } c \text{ is on the interval } (0, 1) \cup (1, 2). \]

b) At what point(s) \( c \) does only the left-hand limit exist?

When \( c = 2 \), only left hand limit exists.

c) At what point(s) \( c \) does only the right-hand limit exist?

When \( c = 0 \), only right hand limit exists.

16. A water balloon dropped from the roof of a small building falls \( y = 4.9t^2 \) m in \( t \) sec. Suppose you wanted to know the speed of the water balloon at exactly \( t = 2 \) seconds. Originally, we used values of \( t \) really close to 2 and found the average rate of change between them. Let’s try something a little different …

a) Instead of using a numeric value “close” to 2, what would be the average speed of the balloon between \( t = 2 \) and \( t = 2 + h \)? (Simplify the expression as much as you can)

\[
\text{Avg. Speed} = \frac{\Delta \text{in dist}}{\Delta \text{in time}} = \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{4.9(2+h)^2 - 4.9(2)^2}{2+h-2} = \frac{4.9[4+4h+h^2] - 4.9(4)}{h} = \frac{4.9(4) + 4.9(4h) + 4.9(h^2) - 4.9(4)}{h} = \frac{19.6 + 4.9h^2}{h}.
\]

b) To find the speed of the balloon at \( t = 2 \), it is tempting to simply plug in \( h = 0 \), however, this yields \( \frac{0}{0} \), which is an “indeterminate form”. We CAN however, evaluate your simplified expression from part a using limit as \( h \to 0 \). Evaluate this limit.

\[
\lim_{h \to 0} \frac{f(2+h) - f(2)}{(2+h) - 2} = \lim_{h \to 0} \frac{f(2)+f(h)}{2+h-2} = \text{indeterminate form}.
\]

After simplifying ...

\[
\lim_{h \to 0} (19.6 + 4.9h) = 19.6 + 4.9(0) = 19.6.
\]

c) Now find and compare the speed of the balloon at \( t = 2 \) like you did earlier in question #4.

As time gets close to \( t = 2 \) sec... \( f(t) = 4.9t^2 \)

\[
\text{avg. speed between } t = 1.5 \text{ and } t = 2 \text{ sec} = \frac{f(2) - f(1.5)}{2 - 1.5} = 17.15
\]

\[
\text{avg. speed between } t = 1.9 \text{ and } t = 2 \text{ sec} = \frac{f(2) - f(1.9)}{2 - 1.9} = 19.11
\]

\[
\text{avg. speed between } t = 1.99 \text{ and } t = 2 \text{ sec} = \frac{f(2) - f(1.99)}{2 - 1.99} = 19.55\text{.}
\]

\[
\text{avg. speed between } t = 1.999 \text{ and } t = 2 \text{ sec} = \frac{f(2) - f(1.999)}{2 - 1.999} = 19.5951
\]

very close!
14a) \( f(x) = \begin{cases} 
4-x & \text{if } x \leq 2 \\
4 & \text{if } x = 2 \\
\frac{x}{2}+3 & \text{if } x > 2 
\end{cases} \)

14b) \( f(x) = \begin{cases} 
1-x^2 & \text{if } x \neq -1 \\
3 & \text{if } x = -1 
\end{cases} \)

15) \( f(x) = \begin{cases} 
\sqrt{1-x^2} & \text{if } 0 \leq x < 1 \\
3 & \text{if } 1 \leq x \leq 2 \\
1 & \text{if } x = 2 
\end{cases} \)
A P Calculus
2.1 Worksheet Day 2

"All work must be shown in this course for full credit. Unsupported answers may receive NO credit."

1. When evaluating limits, what does it mean if direct substitution gives you \( \frac{0}{0} \)? The limit DNE ... (probably due to VA)

2. When evaluating limits, what does it mean if direct substitution gives you \( \frac{0}{0} \)?
   \[
   \frac{0}{0} \text{ means direct subst. tells us nothing} \ldots \text{ Do SOMETHING ELSE!}
   \]

3. What are the methods (options) for dealing with the result \( \frac{0}{0} \)?
   1. Cancel like factors
   2. Rationalize the numerator
   3. Simplify with algebra,

4. Evaluate the following limits algebraically.

   a) \[
   \lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+2)}{(x-3)}
   \]
   \[
   = \lim_{x \to 3} x + 2 = 3 + 2 = 5
   \]

   b) \[
   \lim_{x \to 0} \frac{1}{x} - \frac{1}{x} = \lim_{x \to 0} \frac{2 - 2x}{2(x)} = \lim_{x \to 0} \frac{-x}{4x + 2x}
   \]
   \[
   = \lim_{x \to 0} \frac{-x}{4x + 2x} \cdot \frac{1}{x} = \lim_{x \to 0} \frac{-1}{4 + 2} = \frac{-1}{6}
   \]

   c) \[
   \lim_{x \to -3} \frac{(\sqrt{x^2 + 1}) - 1}{x}
   \]
   \[
   = \lim_{x \to 0} \frac{1}{x} = \lim_{x \to 0} \frac{2x + 1 - 1}{x(\sqrt{2x + 1} + 1)}
   \]
   \[
   = \lim_{x \to 0} \frac{2x}{x(\sqrt{2x + 1} + 1)} = \lim_{x \to 0} \frac{2}{\sqrt{2x + 1} + 1} = \frac{2}{1} = 2
   \]

   d) \[
   \lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4}
   \]
   \[
   = \lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{(x - 4)(\sqrt{x + 5} + 3)}
   \]
   \[
   = \lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{(x - 4)(\sqrt{x + 5} + 3)}
   \]
   \[
   = \lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{(x - 4)(\sqrt{x + 5} + 3)}
   \]

   e) \[
   \lim_{x \to 1} \frac{x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{x - 1}{(x+1)(x-1)}
   \]
   \[
   = \lim_{x \to 1} \frac{1}{x + 1} = \frac{1}{1 + 1} = \frac{1}{2}
   \]

   f) \[
   \lim_{x \to 0} \frac{(4 + x)^2 - 16}{x}
   \]
   \[
   = \lim_{x \to 0} \frac{8x + x^2 - 16}{x}
   \]
   \[
   = \lim_{x \to 0} 8 + x = 8 + 0 = 8
   \]

   g) \[
   \lim_{t \to 2} \frac{t^2 - 3t + 2}{t^2 - 4} = \lim_{t \to 2} \frac{(t-2)(t-1)}{(t+2)(t-2)}
   \]
   \[
   = \lim_{t \to 2} \frac{t-1}{t+2} = \frac{2-1}{4} = \frac{1}{4}
   \]

   h) \[
   \lim_{x \to 0} \frac{(2 + x)^3 - 8}{x} = \lim_{x \to 0} \frac{8x + 12x^2 + 12x + 12}{x}
   \]
   \[
   = \lim_{x \to 0} \frac{8 + 12x + 12}{x} = 0 + 12(0) + 12
   \]
   \[
   = 12
   \]
One of the limits you should know is \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \). This limit ONLY works when the denominator matches the inside of the sine function. If they do not match, you cannot change the inside of a sine function without a trig identity. Your goal will be to correctly show the algebra in order to use this limit.

5. Evaluate each of the following limits analytically. Be sure to show ALL steps in your evaluation.

\[
a) \quad \lim_{x \to 0} \frac{\sin x}{5x} = \lim_{x \to 0} \left( \frac{\sin x}{x} \cdot \frac{1}{5} \right) = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{5} = 1 \cdot \frac{1}{5} = \frac{1}{5}
\]

\[
b) \quad \lim_{x \to 0} \frac{\sin 5x}{5x} = \lim_{x \to 0} \frac{\sin 5x}{5} = \lim_{x \to 0} \sin 5x \cdot \lim_{x \to 0} \frac{5}{5} = \sin 5 \cdot 1 = 1 \cdot 5 = 5
\]

\[
c) \quad \lim_{x \to \frac{\pi}{4}} \frac{\sin(x - \frac{\pi}{4})}{x - \frac{\pi}{4}} = \lim_{x \to \frac{\pi}{4}} \frac{\sin(x - \frac{\pi}{4})}{x - \frac{\pi}{4}} = \left[ \frac{1}{1} \right]
\]

\[
d) \quad \lim_{x \to 0} \frac{3 \sin 4x}{\sin 3x} \cdot \frac{\frac{4x}{3x}}{\frac{\sin 3x}{3x}} = \lim_{x \to 0} \frac{12 \sin 4x}{4x} = \lim_{x \to 0} \frac{4x \sin 4x}{4x} = \frac{4 \cdot 1}{1} = 4
\]

If \( x \to \frac{\pi}{4} \), then \( x - \frac{\pi}{4} \to 0 \)

6. Evaluate each of the following by combining properties of limits and your algebra skills.

\[
a) \quad \lim_{x \to 0} \frac{x + \sin x}{x} = \lim_{x \to 0} \frac{x}{x} + \lim_{x \to 0} \frac{\sin x}{x} = 1 + \lim_{x \to 0} \frac{\sin x}{x} = 1 + 1 = 2
\]

\[
b) \quad \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = 1
\]

\[
c) \quad \lim_{x \to 0} \frac{\sin x}{2x^2 - x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{2x - 1} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{2x - 1} = 1 \cdot \frac{1}{1} = 1
\]

\[
d) \quad \lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{x} = 1 \cdot 1 = 1
\]

7. Consider \( \lim_{x \to 0} \frac{f(x) - f(0)}{x^2} = \)

a) If you use direct substitution, what result do you get? \( \frac{f(0) - f(0)}{0^2} = \frac{b}{0} \) indeterminate form 'Do something else!'

b) Evaluate the limit if \( f(x) = 2x^2 + 1 \). \( \lim_{x \to 0} \frac{\partial x^2 + 1}{x^2} \)

\[
f(0) = \partial (0)^2 + 1
\]

\[
\frac{f(0)}{x^2} = \lim_{x \to 0} \frac{2x^2}{x^2} = \lim_{x \to 0} 2 = 2
\]

8. If \( a \neq 0 \), then \( \lim_{x \to a} \frac{x^2 - a^2}{x^2 - a^2} = \lim_{x \to a} \frac{x^2 - a^2}{(x^2 + a^2)(x - a)} = \lim_{x \to a} \frac{1}{x^2 + a^2} = \frac{1}{a^2 + a^2} = \frac{1}{2a^2} \)
9. Evaluate the following limits analytically (all mixed up):

a) \[ \lim_{x \to 0} \frac{2x - \frac{3}{x}}{x} = \lim_{x \to 0} \frac{\frac{12 - 3(4 + x)}{x} - 4}{x} = \lim_{x \to 0} \frac{-3x}{16} = \frac{-3}{16} \]

b) \[ \lim_{x \to 0} \frac{5x^3 + 8x^2}{3x^4 - 16x} = \lim_{x \to 0} \frac{5x + 8}{3x^2 - 16} = \frac{8}{16} = \frac{1}{2} \]

c) \[ \lim_{x \to 3} \frac{x^2 - 3x}{x} = \lim_{x \to 3} \frac{x(x-3)}{x} = \lim_{x \to 3} (x-3) = 0 - 3 = -3 \]

d) \[ \lim_{x \to 0} \frac{x^2 + 8x^2}{x^2} = \lim_{x \to 0} \frac{9x^2}{x^2} = \frac{9}{1} = 9 \]

e) \[ \lim_{x \to 1} \frac{x}{x^2 - x} = \lim_{x \to 1} \frac{x}{x(x-1)} = \lim_{x \to 1} \frac{1}{x-1} = \text{DNE} \]

f) \[ \lim_{x \to 0} \frac{\sin(2x)}{2x} = \lim_{x \to 0} \frac{2 \cdot \sin(2x)}{2x} = \lim_{x \to 0} 2 \cdot \lim_{x \to 0} \frac{\sin(2x)}{2x} = 2 \cdot 1 = 2 \]

g) \[ \lim_{x \to 0} \frac{\sin(7x)}{3x} = \lim_{x \to 0} \frac{\frac{7}{3} \cdot \sin(7x)}{7x} = \frac{7}{3} \cdot \lim_{x \to 0} \frac{\sin(7x)}{7x} = \frac{7}{3} \cdot 1 = \frac{7}{3} \]

h) \[ \lim_{x \to 1} \frac{x^2 - 5x + 4}{x^2 - 2x - 8} = \lim_{x \to 1} \frac{(x - 4)(x - 1)}{(x - 8)(x + 2)} = \lim_{x \to 1} \frac{4}{12} = \frac{1}{3} \]

12. Evaluate \[ \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x + h)}{h} \]

\[ \therefore \ h \text{ is going to 0 ... not } x \ ... \ so \ treat \ this \ as \ if \ h \ is \ the \ variable \ ... \ your \ final \ answer \ will \ have \ an \ }x \text{ in it.} \]

\[ \lim_{h \to 0} 2x + h = 2x + 0 = 2x \]

13. Suppose \( g(x) = \begin{cases} 2 - x, & \text{if } x \leq 1 \\ \frac{2}{x}, & \text{if } x > 1 \end{cases} \)

a) \[ \lim_{x \to 1^-} g(x) = 2 - 1 = 1 \]

b) \[ \lim_{x \to 1^+} g(x) = \frac{2}{2} = 1 \]

c) \[ \lim_{x \to 1} g(x) = \text{DNE} \]

d) \[ g(1) = 2 - 1 = 1 \]

\[ \lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x) \]
1. Answer the following questions:

a) How do you find horizontal asymptotes? Horizontal asymptotes occur when \( y \to \# \) as \( x \to \pm \infty \). Find them with \( \lim_{x \to \infty} f(x) \) or \( \lim_{x \to -\infty} f(x) \).

b) Which of the parent functions have horizontal asymptotes? List the function(s) and asymptote(s)

\[
\begin{align*}
\text{y} & = \frac{1}{x} \quad \text{HA: y} = 0 \\
\text{y} & = \frac{1}{x^2} \quad \text{HA: y} = 0 \\
\text{y} & = \frac{1}{x^2} \quad \text{HA: y} = 0
\end{align*}
\]

c) How do you find vertical asymptotes? Find the values that make the numerator \( \neq \# \) and denominator \( = 0 \). Find values \( x = a \) that make behavior of \( f(x) \) true for \( x \to a^+ f(x) = \pm \infty \) or \( x \to a^- f(x) = \pm \infty \).

d) Which of the parent functions have vertical asymptotes? List the function(s) and asymptote(s)

\[
\begin{align*}
\text{y} & = \log_b x \quad \text{VA: x} = 0 \\
\text{y} & = \frac{1}{x} \quad \text{VA: x} = 0 \quad \text{y} = \frac{1}{x^2} \quad \text{VA: x} = 0 \\
\text{y} & = \tan x \quad \text{VA: x} = \text{odd mult. of } \frac{\pi}{2}
\end{align*}
\]

e) When must you look for oblique (slanted) asymptotes? How do you find them? Look for SA on a rational function when the degree of numerator > degree of denominator. If there is no HA, look for SA using long division.

2. For each of the following, find (i) \( \lim_{x \to \infty} f(x) \) and (ii) \( \lim_{x \to -\infty} f(x) \). Then (iii) identify all horizontal asymptotes, if any.

\[
\begin{align*}
a) \quad & f(x) = \frac{x-2}{2x^2+3x-5} \quad \text{End Behav. Model:} \quad x \to \infty \\
& f(x) = \frac{4x^3-2x+1}{x^2-2x+1} \quad \text{EBM:} \quad x \to \infty \\
& f(x) = \frac{3x^2-x+5}{x^2-4} \quad \text{EBM:} \quad x \to \infty \\
& f(x) = e^{-x} = \frac{1}{x \cdot e^x} \quad \text{bottom grows faster} \quad f(x) = \frac{1}{x} \quad \text{grow at same rate...} \quad f(x) = \frac{\sin x}{2x^2+x} \quad \text{as } x \to \infty
\end{align*}
\]

d) \( f(x) = e^{-x} = \frac{1}{x \cdot e^x} \quad \text{bottom grows faster} \quad f(x) = \frac{1}{x} \quad \text{grow at same rate...} \quad f(x) = \frac{\sin x}{2x^2+x} \quad \text{as } x \to \infty
\]

e) \( f(x) = \frac{1}{x} \quad \text{bottom grows faster} \quad f(x) = \frac{\sin x}{2x^2+x} \quad \text{as } x \to \infty
\]

3. One of the functions in 2a – 2c has a slanted (oblique) asymptote. Explain why, and then find the asymptote.

2b) may have SA, blc there is no HA and ble. degree of numerator > degree of denominator

\[
\begin{align*}
x^2 - 2x + 1 & = (x + 1) \frac{4x^3 - 0x^2 - 2x + 1}{x^2 - 2x + 1} - (4x^3 - 8x^2 + 4x) \\
\frac{8x^2 + 8}{x^2 - 2x + 1} & = \frac{4x^5 + 0x^2 - 2x + 1}{x^2 - 2x + 1} - (4x^3 - 8x^2 + 4x) \\
\frac{10x - 7}{x^2 - 2x + 1} & = \frac{4x^5 + 0x^2 - 2x + 1}{x^2 - 2x + 1} - (4x^3 - 8x^2 + 4x) \\
\text{SA: } y & = 4x + 8
\end{align*}
\]
4. For each of the following, (i) find the vertical asymptotes of the graph of \( f(x) \) and (ii) describe the behavior of the graph of \( f(x) \) to the left and right of each asymptote.

   a) \( f(x) = \frac{1}{x-3} \)

   i) \( \lim_{x \to 3^-} f(x) = \infty \)
   ii) \( \lim_{x \to 3^+} f(x) = -\infty \)
   iii) \( \lim_{x \to 3^-} f(x) = -\infty \)
   iv) \( \lim_{x \to 3^+} f(x) = \infty \)

   b) \( f(x) = \frac{1}{x^2 - 4} \)

   i) \( \lim_{x \to 2^-} f(x) = \infty \)
   ii) \( \lim_{x \to 2^+} f(x) = -\infty \)
   iii) \( \lim_{x \to -2^-} f(x) = -\infty \)
   iv) \( \lim_{x \to -2^+} f(x) = \infty \)

   c) \( f(x) = \frac{1-x}{2x^2 - 5x - 3} \)

   i) \( \lim_{x \to \frac{1}{2}^-} f(x) = \infty \)
   ii) \( \lim_{x \to \frac{1}{2}^+} f(x) = \infty \)
   iii) \( \lim_{x \to 1^-} f(x) = \infty \)
   iv) \( \lim_{x \to 1^+} f(x) = -\infty \)

5. Find the limit of \( g(x) \) as (i) \( x \to -\infty \), (ii) \( x \to -\infty \), (iii) \( x \to 0^- \), and (iv) \( x \to 0^+ \).

   a) \( g(x) = \begin{cases} 
   \frac{1}{x} & \text{if } x < 0 \\
   \frac{1}{x+1} & \text{if } x \geq 0
   \end{cases} \)

   i) \( \lim_{x \to 0^-} g(x) = 2 \)
   ii) \( \lim_{x \to 0^+} g(x) = \lim_{x \to 0^-} g(x) = -\infty \)

   b) \( g(x) = \begin{cases} 
   \frac{x-2}{x+1} & \text{if } x \leq 0 \\
   \frac{x}{3} & \text{if } x > 0
   \end{cases} \)

   i) \( \lim_{x \to 0^-} g(x) = 0 \)
   ii) \( \lim_{x \to 0^+} g(x) = \lim_{x \to \infty} \frac{2x-3}{x+1} = 3 \)
   iii) \( \lim_{x \to \infty} g(x) = \lim_{x \to 0^-} \frac{2x-3}{x+1} = 3 \)
   iv) \( \lim_{x \to 0^+} g(x) = \lim_{x \to \infty} \frac{1}{x^2} = 0 \)

6. Sketch a function that satisfies the stated conditions. Include any asymptotes.

   a) \( \lim_{x \to -2} f(x) = 1 \)
   b) \( \lim_{x \to -2} f(x) = -1 \)
   c) \( \lim_{x \to \infty} f(x) = 2 \)
   d) \( \lim_{x \to -\infty} f(x) = 2 \)

7. Sketch a function that satisfies the stated conditions. Include any asymptotes.

   a) \( \lim_{x \to 2} f(x) = 1 \)
   b) \( \lim_{x \to 2} f(x) = \infty \)
   c) \( \lim_{x \to \infty} f(x) = 2 \)
8. Explain why there is no value \( L \) for which \( \lim_{x \to \infty} \sin x = L \).

As \( x \to \infty \), \( \sin x \) oscillates between \(-1\) and \(1\). 
\( \therefore \lim_{x \to \infty} \sin x \text{ DNE} \) because a limit is a single, real value.

9. Let \( f(x) = \frac{\cos x}{x} \).

\( a \) Find the domain and range of \( f \).
\( D: (-\infty, 0) \cup (0, \infty) \)
\( R: (-\infty, \infty) \)

\( b \) Is \( f \) even, odd, or neither? Justify your response.
\[ f(-x) = \frac{\cos(-x)}{-x} = \frac{\cos x}{-x} = -\frac{\cos x}{x} = -f(x) \]
Since \( f(-x) = -f(x) \), \( f \) is odd.

\( c \) Find \( \lim_{x \to \infty} f(x) \). Give a reason for your answer.
\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\cos x}{x} = \lim_{x \to \infty} -\frac{\cos x}{x} \]
As \( x \to \infty \), \( \cos x \) oscillates between \(-1\) and \(1\).
As \( x \to \infty \), \( \cos x \) is very large, \( \frac{1}{x} \) is very large\# ... in either case, the fraction approaches 0 because the denominator grows faster.

10. If \( k \) is a positive integer, then \( \lim_{x \to \infty} \frac{x^k}{e^x} = ? \) Explain your answer.

[Try letting \( k = 2 \) ... what about \( k = 10 \)? ... what about \( k = 1000 \)?]

If \( k = \cos \) integer, then \( x^k \) is a polynomial. \( y = e^x \) will always grow faster than a polynomial ... so, the denominator grows faster. \( \therefore \lim_{x \to \infty} \frac{x^k}{e^x} = 0 \)

11. Investigate using tables and graphs to determine the value of each limit:
\[ \lim_{x \to \infty} \frac{3x^2 - 2}{\sqrt{2x^2 + 1}} \quad \text{and} \quad \lim_{x \to \infty} \frac{3x^2 - 2}{\sqrt{2x^2 + 1}} \]
\[ \text{As } x \to \infty, \quad \frac{3x^2 - 2}{\sqrt{2x^2 + 1}} \to \frac{3x}{\sqrt{2x^2 + 1}} = \frac{3x}{\sqrt{2}x} = \frac{3}{\sqrt{2}} \]
\[ \lim_{x \to \infty} \frac{3x^2 - 2}{\sqrt{2x^2 + 1}} = 2.12 \quad \text{or} \quad \lim_{x \to \infty} \frac{3x^2 - 2}{\sqrt{2x^2 + 1}} = 2.12 \]

12. Evaluate each of the following limits using all methods learned from this chapter.

\( a \) \( \lim_{x \to \infty} \left( \frac{2}{x} + 1 \right) \left( \frac{5x^2 - 1}{x^2} \right) = \lim_{x \to \infty} \left( \frac{2}{x} + 1 \right) \cdot \lim_{x \to \infty} \left( \frac{5x^2 - 1}{x^2} \right) = (0 + 1) \cdot 5 = 5 \)

\( b \) \( \lim_{x \to \infty} \left( \frac{5x^2 - 1}{x^2} \right) = \lim_{x \to \infty} \frac{5x^2 - 1}{x^2} \)

\( c \) \( \lim_{x \to \infty} \left( \frac{5x^2 - 1}{x^2} \right) = 5 - \lim_{x \to \infty} \frac{2}{x^2} = 5 - 0 = 5 \)

\( d \) \( \lim_{x \to \infty} \frac{1}{\cos x} = \frac{1}{\cos \frac{\pi}{2}} = \frac{1}{0} \)

\( \therefore \lim_{x \to \frac{\pi}{2}} \sec x \text{ DNE} \) because as \( x \to \frac{\pi}{2} \), \( \sec (x) \) grows without bound.
e) \( \lim_{x \to \infty} e^{-x} \cos x = \lim_{x \to \infty} \frac{\cos x}{e^x} = 0 \)

As \( x \to \infty \), \( e^x \) grows without bound.

f) \( \lim_{x \to \frac{\pi}{2}^-} (2x - \lfloor x \rfloor) = \lim_{x \to \frac{\pi}{4}^-} \frac{2x - \lfloor x \rfloor}{\frac{x}{4}} \)

Check graph or table as \( x \to \frac{x}{2} \) from the right...

\( y = \lfloor 2x - \lfloor x \rfloor \rfloor \) is greatest int. function

\( y-values \) approach 0

h) \( \lim_{n \to \infty} \frac{4n^3}{n^2 + 10000n} = \text{DNE} \) because function increases without bound.

EBM: \( \frac{4n^3}{n^2} = 4n \)

As \( n \to \infty \), \( 4n \to \infty \)

i) \( \lim_{x \to 0} \frac{\sin 2x}{4x} = \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot \frac{1}{2} \)

\( = \lim_{x \to 0} \frac{-1}{2} \cdot \lim_{x \to 0} \frac{1}{2} \)

\( = \left[ \frac{1}{2} \right] \)

k) \( \lim_{x \to 0} \frac{x \sin x + 2 \sin x}{2x^2} = \left[ 0 \right] \)

Can solve either way.

\( = \lim_{x \to 0} \frac{\sin (x+2)}{3x^2} \)

\( = \lim_{x \to 0} \frac{\sin x \cdot x + 2}{2x^2} \)

\( = \lim_{x \to 0} \frac{\sin x}{x} \cdot \pi \cdot \lim_{x \to 0} \frac{x+2}{2x^2} \)

\( = 0 \cdot \frac{1}{2} + 0 \cdot 0 \)

\( = 0 \)

j) \( \lim_{x \to 0} \frac{2x^4 - 2x^2}{x} = \lim_{x \to 0} \frac{2 - \frac{1}{(x+2)}^2}{2(x+2)} \)

\( = \lim_{x \to 0} \frac{2 - \frac{1}{2}x}{2(x+2)} \)

\( = \lim_{x \to 0} \frac{-1}{2(x+2)} \)

\( = \left[ \frac{-1}{2} \right] \)

l) \( \lim_{x \to \infty} \frac{x^2 + 1}{3x^2 - 2x + 5} = \frac{(-2)^2 + 1}{3(-2)^2 - 2(-2) + 5} \)

\( = \left[ \frac{5}{21} \right] \)}
1. What is the definition of continuity (at a point)?
   \[ \lim_{x \to c} f(x) = f(c) \] or \[ \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c). \]

2. Sketch a possible graph for each function described.
   a) \( f(5) \) exists, but \( \lim_{x \to 5} f(x) \) does not exist.
   b) The \( \lim_{x \to 5} f(x) \) exists, but \( f(5) \) does not exist.

3. Use the function \( g(x) \) defined and graphed below to answer the following questions.

   \[ g(x) = \begin{cases} 
   1 & \text{if } -2 < x < -1 \\
   -2x - 1 & \text{if } -1 < x < 0 \\
   1 - x^2 & \text{if } 0 < x < 1 \\
   -2 & \text{if } x = 1 \\
   2x - 2 & \text{if } 1 < x \leq 2 
   \end{cases} \]

   a) Does \( g(1) \) exist? Yes \( g(1) = -2 \)
   b) Does \( \lim_{x \to 1} g(x) \) exist? Yes \( \lim_{x \to 1} g(x) = 0 \)
   c) Does \( \lim_{x \to 1} g(x) = g(1) \)? No
   d) Is \( g \) continuous at \( x = 1 \)?
   \( g(x) \) is not continuous because \( \lim_{x \to 1} g(x) \neq g(1) \)
   e) Is \( g \) defined at \( x = -1 \)?
   No... there is no y-value on graph when \( x = -1 \) and no definition for \( x = -1 \) on piece wise function
   f) Is \( g \) continuous at \( x = -1 \)?
   \( g(x) \) is not continuous because \( g(-1) \) is undefined.
   g) For what values of \( x \) is \( g \) continuous? \( g(x) \) is cont. on \((-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \)
   h) What value should be assigned to \( g(-1) \) to make a new (extended) function continuous at \( x = -1 \)?
   If \( g(-1) = 1 \), then \( g(x) \) would be continuous at \( x = -1 \).
   i) What value should be re-assigned to \( g(1) \) to make \( g \) continuous at \( x = 1 \)?
   Reassign \( g(1) = 0 \) instead of \( g(1) = -2 \)
   j) Is it possible to extend \( g \) to be continuous at \( x = 0 \)? If so, what value should the extended function have there? If not, why not?
   It is not possible to extend \( g \) to be continuous at \( x = 0 \) because \( \lim_{x \to 0^-} g(x) \) will not be \( \lim_{x \to 0} g(x) \) unless we redefine the function on intervals (-1,0) & (0,1).
4. Let \( f(x) = \begin{cases} x^2 - 1 & ; x < 3 \\ 2ax & ; x \geq 3 \end{cases} \). Find a value of \( a \) so that the function \( f \) is continuous.

Using the definition of continuity, justify your response.

If \( f(x) \) is continuous at \( x = 3 \), then

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3).
\]

\[
\lim_{x \to 3^-} x^2 - 1 = \lim_{x \to 3^+} 2ax = 2a(3)
\]

\[
\therefore \quad \text{If } a = \frac{1}{3}, \text{ } f(x) \text{ is continuous at } x = 3.
\]

\[
(3)^2 - 1 = 2a(3)
\]

\[
\Rightarrow \quad 8 = 6a \quad \therefore \quad \boxed{a = \frac{4}{3}}
\]

5. Let \( f(x) = \begin{cases} 2x + 3 & ; x \leq 2 \\ kx + 1 & ; x > 2 \end{cases} \). Find a value of \( k \) so that the function \( f \) is continuous.

Using the definition of continuity, justify your response.

If \( f(x) \) is continuous at \( x = 2 \), then

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2).
\]

\[
\lim_{x \to 2^-} 2x + 3 \quad \lim{kx + 1} = 2(2) + 3
\]

\[
\Rightarrow \quad 2k + 1 = 2(2) + 3
\]

\[
\Rightarrow \quad 2k = 3
\]

\[
\Rightarrow \quad k = \frac{3}{2}
\]

6. Let \( f(x) = \begin{cases} \sqrt{2x^2 - 5x + 7} & ; x < 2 \\ k + 3 & ; x \geq 2 \end{cases} \). Find all values of \( a \) that make \( f \) continuous at 2.

Using the definition of continuity, justify your response.

If \( f(x) \) is continuous at \( x = 2 \), then

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2). \quad \text{must have def. of continuity!}
\]

\[
\lim_{x \to 2^-} (x^2 - a^2x) = \lim_{x \to 2^+} (4 - 2x^2) \quad \text{Same as rt limit}
\]

\[
(x^2 - a^2x) \quad \Rightarrow \quad a = \pm 2 \]

7. If \( f(x) = \begin{cases} \frac{\sqrt{2x^2 + 5x + 7}}{x - 2} & ; x \neq 2 \\ k + 3 & ; x = 2 \end{cases} \), and if \( f \) is continuous at \( x = 2 \), then \( k = ? \)

Using the definition of continuity, justify your response.

If \( f(x) \) is continuous at \( x = 2 \), then

\[
\lim_{x \to 2} f(x) = f(2), \quad \text{must have def of continuity!}
\]

\[
\text{evaluate } \lim_{x \to 2} f(x) \text{ with subst. gives } \frac{1}{0} = k + 3
\]

\[
\text{Do something else!} \quad \boxed{\text{See attached pg}}
\]

8. If the function \( f \) is continuous for all real numbers and if \( f(x) = \frac{x^2 - 4}{x + 2} \), when \( x \neq -2 \), then \( f(-2) = \)

Use the definition of continuity to justify your response.

If \( f \) is continuous for all real #, then \( f \) is also continuous at \( x = -2 \).

\[
\lim_{x \to -2} f(x) = f(-2)
\]

\[
\lim_{x \to -2} \frac{x^2 - 4}{x + 2} = f(-2)
\]

\[
-2 - 2 = f(-2)
\]

\[
-4 = f(-2)
\]

\[
\boxed{f(x) = \begin{cases} \frac{x^2 - 4}{x + 2} & ; x \neq -2 \\ -4 & ; x = -2 \end{cases}}
\]
9. Let \( f \) be the function defined by the following:

\[
\begin{align*}
    f(x) &= \begin{cases} 
        \sin x, & x < 0 \\
        x^2, & 0 \leq x < 1 \\
        2 - x, & 1 \leq x < 2 \\
        x - 3, & x \geq 2 
    \end{cases}
\end{align*}
\]

If \( f(x) \) is discontinuous, \( \text{pts of disc. will occur at } x = 0, x = 1 \) \& \( x = 2 \)

For what values of \( x \) is \( f \) NOT continuous? Use the definition of continuity to explain why.

If \( f(x) \) is cont. at \( x = 0 \), then
\[
\begin{align*}
    \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^+} f(x) = f(0) \\
    \lim_{x \to 0^-} \sin x &= \lim_{x \to 0^+} x^2 = f(0) \\
    0 &= 0^2 = 0^2 \\
    0 &= 0 = 0
\end{align*}
\]

\( \therefore f(x) \) is continuous at \( x = 0 \)

If \( f(x) \) is cont. at \( x = 1 \), then
\[
\begin{align*}
    \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^+} f(x) = f(1) \\
    \lim_{x \to 1^-} x^2 &= \lim_{x \to 1^+} (2 - x) = f(1) \\
    (1)^2 &= 2 - (1) = 2 - 1 \\
    1 &= 1 = 1
\end{align*}
\]

\( \therefore f(x) \) is continuous at \( x = 1 \)

If \( f(x) \) is cont. at \( x = 2 \), then
\[
\begin{align*}
    \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^+} f(x) = f(2) \\
    \lim_{x \to 2^-} (2 - x) &= \lim_{x \to 2^+} (x - 3) = f(2) \\
    2 - 2 &= 2 - 3 = 2 - 3 \\
    0 &= -1 \neq -1
\end{align*}
\]

\( \therefore f(x) \) is not continuous at \( x = 2 \)

10. Write an extended function (see questions 3h and 3i) so that the given function is continuous at the indicated point.

a) \( h(x) = \frac{\sin(5x)}{x} \) at \( x = 0 \)

\[
h(x) = \begin{cases} 
    \sin(5x), & \text{if } x \neq 0 \\
    5, & \text{if } x = 0 
\end{cases}
\]

b) \( k(x) = \frac{x - 4}{\sqrt{x - 2}} \) at \( x = 4 \)

\[
k(x) = \begin{cases} 
    \frac{x - 4}{\sqrt{x - 2}}, & \text{if } x \neq 4 \\
    4, & \text{if } x = 4 
\end{cases}
\]

See attached for work!

11. Multiple Choice: Let \( f \) be the function given by \( f(x) = \frac{(x - 1)(x^2 - 4)}{x^2 - a} \). For what positive values of \( a \) is \( f \) continuous for all real numbers?

\( Bk \) this is multiple choice, only possible values of \( a \) are 1, 2, 4.

A) None \quad B) 1 only \quad C) 2 only \quad D) 4 only \quad E) 1 and 4

If \( a = 1 \), then \( f(x) = \frac{(x - 1)(x^2 - 4)}{(x^2 - 1)} \) ... \( f(x) \) has discontinuity at \( x = -1 \) and \( 1 \) non-removable, removable.

If \( a = 2 \), then \( f(x) = \frac{(x - 1)(x^2 - 4)}{x^2 - 2} \) ... \( f(x) \) has discontinuities at \( x = \pm \sqrt{2} \) non-removable.

If \( a = 4 \), then \( f(x) = \frac{(x - 1)(x^2 - 4)}{x^2 - 4} \) ... \( f(x) \) has discontinuities at \( x = 2 \) and \( x = -2 \) both removable.
12. Let \( g(x) = \frac{x^2 + 5x + 6}{x^2 + 7x + 10} \). \( \Rightarrow g(x) = \frac{(x + 2)(x + 3)}{(x + 2)(x + 5)} \) \( x \neq -2, -5 \)

a) Find the domain of \( g(x) \).

\((-\infty, -5) \cup (-5, -2) \cup (-2, \infty)\)

b) Find the \( \lim_{x \to c} g(x) \) for all values of \( c \) where \( g(x) \) is not defined. \( c = -5, -2 \)

\( \lim_{x \to -5} g(x) = \frac{(-5+2)(-5+3)}{(-5+2)(-5+5)} = \frac{0}{0} \Rightarrow \text{DNE} \)

\( \lim_{x \to -2} g(x) = \lim_{x \to -2} \frac{(x+2)(x+3)}{(x+2)(x+5)} = \frac{-2+3}{-2+5} = \frac{1}{3} \)

because as \( x \to -5 \), \( g(x) \) grows without bound.

\( \lim_{x \to -2} \) \( g(x) \) is not defined.

\( \text{EBN:} \ \frac{x^2}{x^2} = 1 \) \( \lim_{x \to -\infty} g(x) = 1 \) \( \lim_{x \to +\infty} g(x) = 1 \)

\( \therefore g(x) \) has H.A. \( y = 1 \)

d) Find any vertical asymptotes and justify your response.

At \( x = -2 \), \( g(x) \) grows without bound... \( \lim_{x \to -2} g(x) = -\infty \). \( \therefore g(x) \) has V.A. at \( x = -2 \).

C.W.: we already found \( \lim_{x \to -5} g(x) = \frac{1}{3} \); \( g(x) \) has hole at \( x = -5 \).

e) Write an extension to the function so that \( g(x) \) is continuous at \( x = -2 \).

Use the definition of continuity to justify your response.

If \( g(x) \) is continuous at \( x = -2 \), then \( \lim_{x \to -2} g(x) = g(-2) \)

\( g(x) = \begin{cases} \frac{x^2 + 5x + 6}{x^2 + 7x + 10} & \text{if } x \neq -2 \text{ or } -5 \\ \frac{1}{3} & \text{if } x = -2 \end{cases} \)

\( \therefore g(x) \) is cont. if \( \frac{x^2 + 5x + 6}{x^2 + 7x + 10} = \frac{1}{3} \) \( \therefore \) we define \( g(-2) = \frac{1}{3} \)

13. Without using a picture, give a written explanation of why the function \( f(x) = x^2 - 4x + 3 \) has a zero in the interval \([2, 4]\).

\( f(2) = (2)^2 - 4(2) + 3 = -1 \)

Since \( f(x) \) is continuous on \([2, 4]\) and \( f(2) < 0 \) while \( f(4) > 0 \), then \( f(x) = 0 \) somewhere on the interval \((2, 4)\) by the IVT.

14. Without using a picture, give a written explanation of why the function \( f(x) = x^2 + 2x - 3 \) must equal 3 at least once in the interval \([0, 2]\).

\( f(0) = (0)^2 + 2(0) - 3 = -3 \)

Since \( f(x) \) is continuous on \([0, 2]\) and \( f(0) < 3 \) while \( f(2) > 3 \), then \( f(x) = 3 \) at least once on the interval \((0, 2)\) by the IVT.

15. Let \( h(x) = \begin{cases} 3x^2 - 4, & \text{if } x \leq 2 \\ 5 + 4x, & \text{if } x > 2 \end{cases} \).

a) What is \( h(0) \)? \( h(0) = 3(0)^2 - 4 \)

\( h(0) = -4 \)

b) What is \( h(4) \)? \( h(4) = 5 + 4(4) \)

\( h(4) = 21 \)

c) On the interval \([0, 4]\), there is no value of \( x \) such that \( h(x) = 10 \) even though \( h(0) < 10 \) and \( h(4) > 10 \). Explain why this result does not contradict the IVT.

\( h(x) \) is continuous if \( \lim_{x \to 2} h(x) = h(2) \)

\( 3(2)^2 - 4 = 5 + 4(2) = 3(2)^2 - 4 \)

\( 5 = 13 \neq 5 \)

\( \therefore \) Since \( h(x) \) is not continuous on the interval \([0, 4]\), then we cannot use IVT (or contradict).
# 10 \text{a}) \text{ If } h(x) \text{ is continuous at } x = 0, \text{ then } \\
\lim_{x \to 0} \frac{\sin(5x)}{x} = h(0) \\
\lim_{x \to 0} \frac{\sin(5x)}{x} \cdot \frac{5}{5} = h(0) \\
\lim_{x \to 0} \frac{5 \cdot \sin(5x)}{5x} = h(0) \\
5 \cdot 1 = h(0) \\
5 = h(0) \\
\therefore \text{ for } h(x) \text{ to be continuous, we define } h(0) = 5
10b. If \( k(x) \) is continuous at \( x = 4 \), then

\[
\lim_{x \to 4} k(x) = k(4)
\]

\[
\lim_{x \to 4} \frac{x-4}{\sqrt{x} - 2} = k(4)
\]

\[
\lim_{x \to 4} \frac{(x-4)(\sqrt{x} + 2)}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = k(4)
\]

\[
\lim_{x \to 4} \frac{(x-4)(\sqrt{x} + 2)}{x - 4} = k(4)
\]

\[
\lim_{x \to 4} \sqrt{x} + 2 = k(4)
\]

\[
\sqrt{4} + 2 = k(4)
\]

\[
4 = k(4)
\]

\[
\therefore \text{for } k(x) \text{ to be continuous, we define } k(4) = 4
\]